L^∞ estimates and integrability by compensation in Besov-Morrey spaces and applications

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Abstract

L^∞ estimates in the integrability by compensation result of H. Wente ([26]) fail in dimension larger than two when Sobolev spaces are replaced by the ad-hoc Morrey spaces (in dimension $n \geq 3$).
However, in this paper we prove that L^∞ estimates hold in arbitrary dimension when Morrey spaces are replaced by their Littlewood Paley counterparts: Besov-Morrey spaces.
As an application we prove the existence of conservation laws to solution of elliptic systems of the form

$$-\Delta u = \Omega \cdot \nabla u$$

where Ω is antisymmetric and both ∇u and Ω belong to these Besov-Morrey spaces for which the system is critical.

1 Introduction

In this section we will give the precise statement of our results and add some remarks.
For the sake of simplicity, in what follows we will use the abbreviation a_x for $\frac{\partial}{\partial x}a$.

Our work was motivated by Rivière's [14] article about Schrödinger systems with antisymmetric potentials, i.e. systems of the form

$$-\Delta u = \Omega \cdot \nabla u \tag{1}$$

with $u \in W^{1,2}(\omega, \mathbb{R}^m)$ and $\Omega \in L^2(\omega, so(m) \otimes \Lambda^1\mathbb{R}^n)$, $\omega \subset \mathbb{R}^n$.
The differential equation (1) has to be understood in the following sense:
For all indices $i \in \{1, \dots, m\}$ we have $-\Delta u^i = \sum_{j=1}^m \Omega^i_j \cdot \nabla u^j$ and $L^2(\omega, so(m) \otimes \Lambda^1\mathbb{R}^n)$ means that $\forall\, i, j \in \{1, \dots, m\}$, $\Omega^i_j \in L^2(\omega, \Lambda^1\mathbb{R}^n)$ and $\Omega^i_j = -\Omega^j_i$.
In particular, it was the result that in dimension $n = 2$ solutions to (1) are continuous which attracted our interest.

The interest for such systems originates in the fact that they "encode" all Euler-Lagrange equations for conformally invariant quadratic Lagrangians in dimension 2 (see [14] and also [9]).

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In what follows we will take $\omega = B_1^n(0)$, the n-dimensional unit ball.

In the above cited work, there were three crucial ideas:

- **Antisymmetry of Ω**
  If we drop the assumption that Ω is symmetric, there may occur solutions which are not continuous as the following example shows:
  Let $n = 2$, $u^i = 2\log\log\frac{1}{r}$ for $i = 1, 2$ and let

$$\Omega = \begin{pmatrix} \nabla u^1 & 0 \\ 0 & \nabla u^2 \end{pmatrix}$$

  Obviously, u satisfies equation (1) with the given Ω but is not continuous.

- **Construction of conservation laws**
  In fact, once there exist $A \in L^\infty(B_1^n(0), M_m(\mathbb{R})) \cap W^{1,2}(B_1^n(0), M_m(\mathbb{R}))$ such that

$$d^*(dA - A\Omega) = 0. \tag{2}$$

  for given $\Omega \in L^2(B_1^n(0), so(m) \otimes \Lambda^1\mathbb{R}^n)$, then any solution u of (1) satisfies the following conservation law

$$d(*Adu + (-1)^{n-1}(*B) \wedge du) = 0 \tag{3}$$

  where B satisfies $-d^*B = dA - A\Omega$.
  The existence of such an A (and B) is proved by Rivière in [14] and relies on a **non linear Hodge decomposition** which can also be interpreted as a **change of gauge**. (see in our case theorem 1.5)

- **Understanding the linear problem**
  The proof of the above mentioned regularity result uses the result below for the linear problem:

  Theorem 1.1 *([26],[7], [24])*
  Let a, b satisfy $\nabla a, \nabla b \in L^2$ and let φ be the unique solution to

$$\begin{cases} -\Delta\varphi = \nabla a \cdot \nabla^\perp b = *(da \wedge db) = a_x b_y - a_y b_x \text{ in } B_1^n(0) \\ \varphi = 0 \text{ on } \partial B_1^n(0). \end{cases} \tag{4}$$

  Then φ is continuous and it holds that

$$||\varphi||_\infty + ||\nabla\varphi||_2 + ||\nabla^2\varphi||_1 \leq C||\nabla a||_2\,||\nabla b||_2. \tag{5}$$

  Note that the L^∞ estimate in (5) is the key point for the existence of A, B satisfying (2).

A more detailed explanation of these key points and their interplay can be found in Rivière's overview [15].

Our strategy to extend the cited regularity result to domains of arbitrary dimension is to find first of all a good generalisation of Wente's estimate. Here, the first question is to detect a suitable substitute for L^2 since obviously for $n \geq 3$ from the fact that $a, b \in W^{1,2}$ we can not conclude that φ is continuous.

So we have to reduce our interest to a smaller space than L^2. A first idea is to look at the Morrey space $\mathcal{M}_2^n$, i.e. at the spaces of all functions $f \in L^2_{loc}(\mathbb{R}^n)$ such that

$$||f|\mathcal{M}_2^n|| = \sup_{x_0 \in \mathbb{R}^n} \sup_{R>0} R^{1-n/2} ||f|L^2(B(x_0,R))|| < \infty.$$

The choice of this space was motivated by the following observation (for details see [16]):
For stationary harmonic maps u we have the following monotonicity estimate

$$r^{2-n} \int_{B_r^n(x_0)} |\nabla u|^2 \leq R^{2-n} \int_{B_R^n(x_0)} |\nabla u|^2$$

for all $r \leq R$. From this, it is rather natural to look at the Morrey space $\mathcal{M}_2^n$.

Unfortunately, this first try is not successful as the following counterexample in dimension $n = 3$ shows:
Let $a = \frac{x_1}{|x|}$ and $b = \frac{x_2}{|x|}$. As required $\nabla a, \nabla b \in \mathcal{M}_2^3(B_1^3(0))$. The results in ([7]) imply that the unique solution φ of (4) satisfies $\nabla^2 \varphi \in \mathcal{M}_1^{\frac{3}{2}}$ but φ is not bounded!
Therefore, in [17] the attempt to construct conservation laws for (1) in the framework of Morrey spaces fails.

Another drawback is that C^∞ is not dense in $\mathcal{M}_2^n$. This point is particularly important if one has in mind the proof via paraproducts of Wente's L^∞ bound for the solution φ.
In this paper we shall study L^∞ estimates by replacing the Morrey spaces $\mathcal{M}_2^n$ by their "nearest" Littlewood Paley counterpart, the Besov-Morrey spaces $B^0_{\mathcal{M}_2^n,2}$, i.e. the spaces of $f \in \mathcal{S}'$ such that

$$\Big(\sum_{j=0}^{\infty} ||\mathcal{F}^{-1}\varphi_j \mathcal{F} f|\mathcal{M}_2^n(\mathbb{R}^n)||^2 \Big)^{\frac{1}{2}} < \infty$$

where $\varphi = \{\varphi_j\}_{j=0}^{\infty}$ is a suitable partition of unity.
It turns out that we have a suitable density result at hand, see lemma 2.14. These spaces were introduced by Kozono and Yamazaki in [10] and applied to the study of the Cauchy problem for the Navier-Stokes equation and semilinear heat equation (see also [12]).

Note, that we have the following **natural embeddings**: $B^0_{\mathcal{M}_2^n,2} \subset \mathcal{M}_2^n$ (see lemma 2.10) and on compact subsets $B^0_{\mathcal{M}_2^n,2}$ is a natural subset of L^2 (see lemma 2.13).

The success to which these Besov-Morrey spaces give rise relies crucially on the fact that **we first integrate and then sum**!
In the spirit of the scales of Triebel-Lizorkin and Besov spaces (definition are restated in the next section) where we have for $0 < q \leq \infty$ and $0 < p < \infty$

$$B^s_{p,\min\{p,q\}} \subset F^s_{p,q} \subset B^s_{\max\{p,q\}}$$

and due to the fact that for $1 < q \le p < \infty$

$$||f||_{\mathcal{M}_q^p} \simeq \Big|\Big|\Big(\sum_{j=0}^{\infty} |\mathcal{F}^{-1}\varphi_j \mathcal{F} f|^2\Big)^{\frac{1}{2}}\Big|\Big|_{\mathcal{M}_q^p}$$

it is obvious to exchange the order of summability and integrability in order to find a smaller space starting from a given one.

A more detailed exposition of the framework of Besov-Morrey spaces is given in the next section.

We have

Theorem 1.2 *i) Assume that $a, b \in B^0_{\mathcal{M}_2^n,2}$, and assume further that*

$$a_x, a_y, b_x, b_y \in B^0_{\mathcal{M}_2^n,2} \textit{ where } x, y = z_i, z_j \textit{ with } i, j, \in \{1, \ldots, n\}\,.$$

Then any solution of

$$-\Delta u = a_x b_y - a_y b_x$$

is continuous and bounded.

ii) Assume that a_x, a_y, b_x and b_y are distributions whose support is contained in $B_1^n(0)$ and belong to $B^0_{\mathcal{M}_2^n,2}$, $n \ge 3$.
Moreover, let u be a solution (in the sense of distributions) of

$$-\Delta u = a_x b_y - b_x a_y.$$

Then it holds

$$\nabla u \in B^0_{\mathcal{M}_2^n,1}.$$

iii) Assume that a_x, a_y, b_x and b_y are distributions whose support in $B_1^n(0)$ and belong to $B^0_{\mathcal{M}_2^n,2}$.
Moreover, let u be a solution (in the sense of distributions) of

$$-\Delta u = a_x b_y - b_x a_y.$$

Then it holds

$$\nabla^2 u \in B^{-1}_{\mathcal{M}_2^n,1} \subset B^{-2}_{\infty,1}.$$

Remark 1.3

- If we reduce our interest to dimension $n = 2$, our assumption in the theorem below coincide with the original ones in Wente's framework due to the fact that $\mathcal{M}_2^2 = L^2$ and $B^0_{2,2} = L^2 = F^0_{2,2}$.
- Obviously we have the a-priori bound

$$||u||_\infty \le C\big(||a|B^0_{\mathcal{M}_2^n,2}|| + ||\nabla a|B^0_{\mathcal{M}_2^n,2}||\big)\big(||b|B^0_{\mathcal{M}_2^n,2}|| + ||\nabla b|B^0_{\mathcal{M}_2^n,2}||\big).$$

- Now, if we use a homogeneous partition of unity instead of an inhomogeneous as before, our result holds if we replace the spaces $B^0_{\mathcal{M}_2^n,2}$ by the spaces $\mathcal{N}^0_{n,2,2}$. For further information about these homogeneous function spaces we refer to Mazzucato's article [12].

- Note that the estimate $\nabla u \in B^0_{\mathcal{M}^n_2,1}$ implies that u is bounded and continuous.

As an application of what we did so far, we would like to present an adaptation of Rivière's construction of conservation laws via gauge transformation (see [14]) to our setting, more precisely we are able to prove the following assertion:

Theorem 1.4 *Let $n \geq 3$. There exist constants $\varepsilon(m) > 0$ and $C(m) > 0$ such that for every $\Omega \in B^0_{\mathcal{M}^n_2,2}(B^n_1(0), so(m) \otimes \Lambda^1\mathbb{R}^n)$ which satisfies*

$$||\Omega|B^0_{\mathcal{M}^n_2,2}|| \leq \varepsilon(m)$$

there exist $A \in L^\infty(B^n_1(0), Gl_m(\mathbb{R})) \cap B^1_{\mathcal{M}^n_2,2}$ and $B \in B^1_{\mathcal{M}^n_2,2}(B^n_1(0), M_m(\mathbb{R}) \otimes \Lambda^2\mathbb{R}^n)$ such that

i)

$$d_\Omega := dA - A\Omega = -d^*B = - * d * B$$

ii)

$$||\nabla A|B^0_{\mathcal{M}^n_2,2}||+||\nabla A^{-1}|B^0_{\mathcal{M}^n_2,2}||+\int_{B^n_1(0)} ||dist(A, SO(m))||^2_\infty \leq C(M)||\Omega|B^0_{\mathcal{M}^n_2,2}||$$

iii)

$$||\nabla B|B^0_{\mathcal{M}^n_2,2}|| \leq C(m)||\Omega|B^0_{\mathcal{M}^n_2,2}||.$$

This finally leads to the following regularity result:

Corollary 1.5 *Let the dimension n satisfy $n \geq 3$. Let $\varepsilon(m)$, Ω, A and B be as in theorem 1.4. Then any solution u of*

$$-\Delta u = \Omega \cdot \nabla u$$

satisfies the conservation law

$$d(*Adu + (-1)^{n-1}(*B) \wedge du) = 0.$$

Moreover, any distributional solution of $\Delta u = -\Omega \cdot \nabla u$ which satisfies in addition

$$\nabla u \in B^0_{\mathcal{M}^n_2,2}$$

is continuous.

Remark 1.6 Note that the continuity assertion of the above corollary is already contained in [16], but our result differs from [16] (see also [18] for a modification of the proof of Rivière and Struwe) in so far, as on one hand we do not impose any smallness of the norm of the gradient of a solution and really construct A and B (see theorem 1.4) and not only construct Ω and ξ such that $P^{-1}dP + P^{-1}\Omega P = *d\xi$, but on the other hand work in a slightly smaller space.

The present article is organised as follows: After recalling some basic definitions and preliminary facts in section 2 we give in the third section the proofs of the statements claimed before.

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2 Definitions and preliminary results

We recall the important definitions and state basic results we will use.

2.1 Besov and Triebel-Lizorkin spaces

2.1.1 Non-homogeneous Besov and Triebel-Lizorkin spaces

In order to define them we have to introduce some additional notions:

Definition 2.1 ($\mathbf{\Phi(\mathbb{R}^n)}$) *Let $\mathbf{\Phi}(\mathbb{R}^{\mathbf{n}})$ be the collection of all systems $\varphi = \{\varphi_j(x)\}_{j=0}^{\infty} \subset \mathcal{S}(\mathbb{R}^n)$ such that*

$$\begin{cases} supp\ \varphi_0 \subset \{x|\ |x| \leq 2\} \\ supp\ \varphi_j \subset \left\{x|\ 2^{j-1} \leq |x| \leq 2^{j+1}\right\} \quad \text{if } j = 1, 2, 3, \ldots, \end{cases}$$

for every multi-index α there exists a positive number C_α such that

$$2^{j|\alpha|}|D^\alpha \varphi_j(x)| \leq C_\alpha \text{ for all } j = 1, 2, 3, \ldots \text{ and all } x \in \mathbb{R}^n$$

and

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1\ \forall x \in \mathbb{R}^n$$

Remark 2.2

- Note that in the above expression $\sum_{j=0}^{\infty} \varphi_j(x) = 1$ the sum is locally finite!
- Example of a system φ which belongs to $\Phi(\mathbb{R}^n)$:
  We start with an arbitrary $C_0^\infty(\mathbb{R}^n)$ function ψ which has the following properties: $\psi(x) = 1$ for $|x| \leq 1$ and $\psi(x) = 0$ for $|x| \geq \frac{3}{2}$. We set $\varphi_0(x) = \psi(x)$, $\varphi_1(x) = \psi(\frac{x}{2}) - \psi(x)$, and $\varphi_j(x) = \varphi_1(2^{-j+1}x)$, $j \geq 2$. Then it is easy to check that this family φ satisfies the requirements of our definition.
  Moreover, we have $\sum_{j=0}^{n} \varphi_j(x) = \psi(2^{-n}x)$, $n \geq 0$.
  By the way, other examples of $\varphi \in \Phi$, apart from this one, can be found in [17], [25] or [6].)

Now, we can state the definitions of the above mentioned Besov and Triebel-Lizorkin spaces.

Definition 2.3 (Besov spaces and Triebel-Lizorkin spaces) *Let $-\infty < s < \infty$, let $0 < q \leq \infty$ and let $\varphi \in \Phi(\mathbb{R}^n)$.*

i) If $0 < p \leq \infty$ then the (non-homogeneous) Besov spaces $\mathbf{B^s_{p,q}}(\mathbb{R}^{\mathbf{n}})$ consist of all $f \in \mathcal{S}'$ such that the following inequality holds

$$||f|B^s_{p,q}(\mathbb{R}^n)||^{\varphi} = ||2^{js}\mathcal{F}^{-1}\varphi_j\mathcal{F}f|l^q(L^p(\mathbb{R}^n))|| < \infty$$

ii) If $0 < p < \infty$ then the (non-homogeneous) Triebel-Lizorkin spaces $\mathbf{F^s_{p,q}}(\mathbb{R}^{\mathbf{n}})$ consist of all $f \in \mathcal{S}'$ such that the following inequality holds

$$||f|F^s_{p,q}(\mathbb{R}^n)||^{\varphi} = ||2^{js}\mathcal{F}^{-1}\varphi_j\mathcal{F}f|L^p(\mathbb{R}^n, l^q)|| < \infty$$

iii) If $p = \infty$ then the spaces $\mathbf{F^s_{\infty,q}}(\mathbb{R}^\mathbf{n})$ consist of all $f \in \mathcal{S}'$ such that $\exists \{f_k(x)\}_{k=0}^\infty \subset L^\infty(\mathbb{R}^n)$ such that the following holds

$$f = \sum_{k=0}^{\infty} \mathcal{F}^{-1}\varphi_k \mathcal{F} f_k \text{ in } \mathcal{S}'(\mathbb{R}^n)$$

and

$$||2^{sk} f_k | L^\infty(\mathbb{R}^n, l^q)|| < \infty.$$

Moreover we set

$$||f|F^s_{\infty,q}(\mathbb{R}^n)||^\varphi = \inf ||2^{sk} f_k | L^\infty(\mathbb{R}^n, l^q)||$$

where the infimum is taken over all admissible representations of f.

Here $\mathcal{F}$ denotes the Fourier transform and

$$||f_k|l^q(L^p(\mathbb{R}^n))|| = \left(\sum_{k=0}^{\infty} \big(\int |f_k(x)|^p dx \big)^{\frac{q}{p}} \right)^{\frac{1}{q}}$$

and

$$||f_k|L^p(\mathbb{R}^n, l^q)|| = \left(\int \big(\sum_{k=0}^{\infty} |f_k(x)|^q \big)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}.$$

Recall that the spaces $B^s_{p,q}$ and $F^s_{p,q}$ are independent of the choice of φ (see [25]).

Most of the important fact (embeddings, relation with other function spaces, multiplier assertions and so on) about these spaces can be found in [17] and [25]. In what follows we will give precise indications where a result we use is proved.

2.1.2 Besov-Morrey spaces

In stead of combining L^p-norms ans l^q-norm one can also combine Morrey- (respectively Morrey-Campanato-) norms with l^q-norms. This idea was first introduced and applied by Kozono and Yamazaki in [10].
In order to make the whole notation clear and to avoid misunderstanding we will recall some definitions.
We start with the definition of Morrey spaces

Definition 2.4 (Morrey spaces) *Let $1 \leq q \leq p < \infty$.*

i) The Morrey spaces $\mathcal{M}^\mathbf{p}_\mathbf{q}(\mathbb{R}^\mathbf{n})$ consist of all $f \in L^q_{loc}(\mathbb{R}^n)$ such that

$$||f|\mathcal{M}^p_q|| = \sup_{x_0 \in \mathbb{R}^n} \sup_{R>0} R^{n/p-n/q} ||f|L^q(B(x_0,R))|| < \infty$$

ii) The local Morrey spaces $\mathbf{M}^\mathbf{p}_\mathbf{q}(\mathbb{R}^\mathbf{n})$ consist of all $f \in L^q_{loc}(\mathbb{R}^n)$ such that

$$||f|\mathcal{M}^p_q|| = \sup_{x_0 \in \mathbb{R}^n} \sup_{0<R\leq 1} R^{n/p-n/q} ||f|L^q(B(x_0,R))|| < \infty$$

where $B(x_0, R)$ denotes the closed ball in $\mathbb{R}^n$ with center x_0 and radius R.

Note that it is easy to see that the spaces $\mathcal{M}^p_q$ and M^p_q coincide on compactly supported functions.
Apart from these spaces of regular distributions, i.e. function belonging to L^1_{loc}, in the case $q = 1$ we are even allowed to look at measures in stead of functions. More precisely we have the following measure spaces of Morrey type. They will become useful later on in a rather technical context.

Definition 2.5 (Measure spaces of Morrey type) *Let $1 \leq p < \infty$.*

i) The measure spaces of Morrey type $\mathcal{M}^{\mathbf{p}}(\mathbb{R}^{\mathbf{n}}) = \mathcal{M}^p$ consist of all Radon measures μ such that

$$||\mu|\mathcal{M}^p|| = \sup_{x_0 \in \mathbb{R}^n} \sup_{R>0} R^{n/p-n} |\mu|(B(x_0, R)) < \infty.$$

ii) The local measure spaces of Morrey type $\mathbf{M_q}(\mathbb{R}^{\mathbf{n}}) = M^p$ consist of all Radon measures μ such that

$$|\mu|\mathcal{M}^p|| = \sup_{x_0 \in \mathbb{R}^n} \sup_{0<R\leq 1} R^{n/p-n} |\mu|(B(x_0, R)) < \infty$$

where as above $B(x_0, R)$ denotes the closed ball in $\mathbb{R}^n$ with center x_0 and radius R.

Remember that all the spaces we have seen so far, i. e. $\mathcal{M}^p_q$, M^p_q, $\mathcal{M}^p$ and M^p are Banach spaces with the norms indicated before. Moreover, $\mathcal{M}^p_1$ and M^p_1 can be considered as closed subspaces of $\mathcal{M}^p$ and M^p respectively, consisting of all those measures which are absolutely continuous with respect to the Lebesgue measure.
For details, see e.g. [10].
Once we have the above definition of Morrey spaces (of regular distributions), we now define the Besov-Morrey spaces in the same way as we constructed the Besov spaces, of course with the necessary changes.

Definition 2.6 (Besov-Morrey spaces) *Let $1 \leq q \leq p < \infty$, $1 \leq r \leq \infty$ and $s \in \mathbb{R}$.*

i) Let $\varphi \in \dot{\Phi}(\mathbb{R}^n)$. The homogeneous Besov-Morrey spaces $\mathcal{N}^{\mathbf{s}}_{\mathbf{p},\mathbf{q},\mathbf{r}}$ consist of all $f \in \mathcal{Z}'$ such that

$$||f|\mathcal{N}^s_{p,q,r}(\mathbb{R}^n)||^{\varphi} = \Big(\sum_{j=-\infty}^{\infty} 2^{jsr} ||\mathcal{F}^{-1}\varphi_j \mathcal{F} f|\mathcal{M}^p_q(\mathbb{R}^n)||^r \Big)^{\frac{1}{r}} < \infty.$$

ii) Let $\varphi \in \Phi(\mathbb{R}^n)$. The inhomogeneous Besov-Morrey spaces $\mathbf{N}^{\mathbf{s}}_{\mathbf{p},\mathbf{q},\mathbf{r}}$ consist of all $f \in \mathcal{S}'$ such that

$$||f|N^s_{p,q,r}(\mathbb{R}^n)||^{\varphi} = \Big(\sum_{j=0}^{\infty} 2^{jsr} ||\mathcal{F}^{-1}\varphi_j \mathcal{F} f|M^p_q(\mathbb{R}^n)||^r \Big)^{\frac{1}{r}} < \infty.$$

Note that since $L^p(\mathbb{R}^n) = \mathcal{M}^p_p(\mathbb{R}^n)$ the framework of the $\mathcal{N}^s_{p,q,r}(\mathbb{R}^n)$ can be seen as a generalisation of the framework of the homogeneous Besov spaces.
In our further work we will crucially use still another variant of spaces which are defined via Paley-Littlewood decomposition. We will use the decomposition into frequencies of positive power but measure the single contributions in a homogeneous Morrey norm:

Definition 2.7 (The spaces $B^s_{\mathcal{M}^p_q,r}$) *i) Let $1 \le q \le p < \infty$, $1 \le r \le \infty$ and $s \in \mathbb{R}$. Let $\varphi \in \Phi(\mathbb{R}^n)$. The spaces $\mathbf{B^s_{\mathcal{M}^p_q,r}}$ consist of all $f \in \mathcal{S}'$ such that*

$$||f|B^s_{\mathcal{M}^p_q,r}(\mathbb{R}^n)||^{\varphi} = \Big(\sum_{j=0}^{\infty} 2^{jsr}||\mathcal{F}^{-1}\varphi_j\mathcal{F}f|\mathcal{M}^p_q(\mathbb{R}^n)||^r\Big)^{\frac{1}{r}} < \infty.$$

ii) The spaces $\mathbf{B^s_{\mathcal{M}^p_q,r}}(\Omega)$ where Ω is a bounded domain in $\mathbb{R}^n$ consist of all $f \in B^s_{\mathcal{M}^p_q,r}$ which in addition have compact support contained in Ω.

Remark 2.8

i) Again, as in the case of Besov and Triebel-Lizorkin spaces, all the spaces defined above do not depend on the choice of φ.

ii) Previously we mentioned that our interest in these latter spaces was motivated by the work of Rivière and Struwe (see [17]) let us say a few words about this. In [17] the authors used the homogeneous Morrey space $L^{2,n-2}_1$ with norm

$$||f||^2_{L^{2,n-2}_1} = \sup_{x_0\in\mathbb{R}^n}\sup_{r>0}\Big(\frac{1}{r^{n-2}}\int_{B-r(x_0)}|\nabla u|^2\Big).$$

Note that $u \in L^{2,n-2}_1$ is equivalent to the fact that for all radii $r > 0$ and all $x_0 \in \mathbb{R}^n$ we have the inequality

$$||\nabla u||_{L^2(B_r(x_0))} \le Cr^{(n-2)/p} = Cr^{\frac{n}{2}-\frac{2}{2}}$$

but this latter estimate is again equivalent to the fact that $\nabla u \in \mathcal{M}^n_2$. Finally we remember that $\mathcal{M}^n_2 = \mathcal{N}^0_{n,2,2}$ (see for instance [12]) and note that $\nabla u \in \mathcal{N}^0_{n,2,2}$ is equivalent to $u \in \mathcal{N}^1_{n,2,2}$ since for all s - even for the negative ones - we have the equivalence $2^s||u^s||_{\mathcal{M}^n_2} \simeq ||(\nabla u)^s||_{\mathcal{M}^n_2}$ because we always avoid the origin in the Fourier space and also near the origin work with annuli with radii $r \simeq 2^s$.

Before we continue, let us state a few facts concerning the spaces $B^s_{\mathcal{M}^p_q,r}$ which are interesting and important.

Lemma 2.9 *i) The spaces $B^s_{\mathcal{M}^p_q,r}$ are complete for all possible choices of indices.*

ii) a) Let $s > 0$, $1 \le q \le p < \infty$, $1 \le r \le \infty$ and $\lambda > 0$. Then

$$||f(\lambda\cdot)|B^s_{\mathcal{M}^p_q,r}|| \le C\lambda^{-\frac{n}{p}}\sup\{1,\lambda\}^s\,||f|B^s_{\mathcal{M}^p_q,r}||.$$

b) Let $s = 0$, $1 \le q \le p < \infty$, $1 \le r \le \infty$ and $\lambda > 0$. Then

$$||f(\lambda\cdot)|B^s_{\mathcal{M}^p_q,r}|| \le C\lambda^{-\frac{n}{p}}(1+|\log\lambda|)^{\alpha}||f|B^s_{\mathcal{M}^p_q,r}||$$

where

$$\alpha = \frac{1}{r}\ \textit{if}\ \lambda > 1\ \ \textit{and}\ \alpha = 1-\frac{1}{r} = \frac{1}{r'}\ \textit{if}\ 0 < \lambda < 1.$$

The first assertion is obtained by the same proof as the corresponding claim for the spaces $N^s_{p,q,r}$ in [10].
The second fact is a variation of a well known proof given in [5].

Furthermore we have the following embedding result which relates the spaces $B^0_{\mathcal{M}^p_q,r}$ to the Morrey spaces with the same indices respectively, similar for the spaces $N^0_{p,q,r}$.

Lemma 2.10 *Let $1 < q \leq 2$, $1 < q \leq p < \infty$ and $r \leq q$. Then*

$$B^0_{\mathcal{M}^p_q,r} \subset \mathcal{M}^p_q$$

and

$$N^0_{p_q,r} \subset M^p_q.$$

From this result we immediately deduce the following corollary.

Corollary 2.11 *Let $1 < q \leq 2$, $1 < q \leq p < \infty$ and $r \leq q$ and assume that $f \in B^0_{\mathcal{M}^p_q,r}$ has compact support. Then $f \in L^q$.*

This holds because of the preceding lemma and the fact that for a bounded domain Ω we have the embedding $M^p_q(\Omega) \subset L^q(\Omega)$.
Similar to the result that $W^{1,p} = F^1_{p,2}$, $1 < p < \infty$ we have the following lemma.

Lemma 2.12 *Assume that f is a compactly supported distribution. Then, if $1 < q \leq 2$, $1 < q \leq p < \infty$ and $r \leq q$, the following two norms are equivalent*

$$||f|B^0_{\mathcal{M}^p_q,r}|| + ||\nabla f|B^0_{\mathcal{M}^p_q,r}||$$

$$||f|B^1_{\mathcal{M}^p_q,r}||.$$

Moreover, also the fact that for a compactly supported distribution the homogeneous and the inhomogeneous Sobolev norms are equivalent, we have the following result.

Lemma 2.13 *Let $1 < q \leq 2$, $1 < q \leq p < \infty$, $2 \leq p$, $r \leq q$ and $n \geq 3$. Assume that the distribution f has the following properties: f has compact support and $\nabla f \in B^0_{\mathcal{M}^p_q,r}$. Then*

$$f \in B^1_{\mathcal{M}^p_q,r}.$$

As a by-product of our studies we have the following density result.

Lemma 2.14 *Let $1 \leq q \leq p < \infty$, $1 \leq r \leq \infty$ and $s \in \mathbb{R}$. Then O_M is dense in $N^s_{p,q,r}$ respectively in $\mathcal{N}^s_{p,q,r}$ and $B^s_{\mathcal{M}^p_q,r}$ where O_M denotes the space of all C^∞-functions such that $\forall \beta \in \mathbb{N}^n$ there exist constants $C_\beta > 0$ and $m_\beta \in \mathbb{N}$ such that*

$$|\partial^\beta f(x)| \leq C_\beta (1 + |x|)^{m_\beta} \; \forall x \in \mathbb{R}^n.$$

Moreover, if $f \in N^s_{p,q,r}$ or $f \in B^s_{\mathcal{M}^q_p r,}$ with $s \geq 0$, $1 \leq q \leq 2$ and $1 \leqq \leq p \leq \infty$ has compact support, it can be approximated by elements in C^∞_0.

Last, but not least we would like to mention a stability result which we will apply later on.

Lemma 2.15 *Let* $g \in B^0_{\mathcal{M}^n_2,2}$ *and* $f \in B^1_{\mathcal{M}^n_2,2,} \cap L^\infty$. *Then*

$$||gf|B^0_{\mathcal{M}^n_2,2}|| \leq C||g|B^0_{\mathcal{M}^n_2,2}||(||f|B^1_{\mathcal{M}^n_2,2}|| + ||f||_\infty),$$

i.e. $B^0_{\mathcal{M}^n_2,2}$ *is stable under multiplication with a function in* $B^1_{\mathcal{M}^n_2,2} \cap L^\infty$.

The proofs of lemma 2.10, 2.12, 2.13, 2.14 and 2.15 are given in the next section.

For further information about the Besov-Morrey spaces, see [10], [11] and [12].

2.2 Spaces involving Choquet integrals

In what follows, we will use a certain description of the pre-dual space of $\mathcal{M}^1$. Before we can state this assertion we have to introduce some function spaces involving the so-called Choquet integral. A general reference for this section is [1] and the references given therein.
We start with the notion of Hausdorff capacity:

Definition 2.16 (Hausdorff capacity) *Let* E *be a subset of* $\mathbb{R}^n$ *and let* $\{B_j\}$, $j = 1, 2, \dots$ *be a cover of* E*, i.e.* $\{B_j\}$ *is a countable collection of open balls* B_j *with radius* r_j *such that* $E \subset \cup_j B_j$. *Then we define the* ***Hausdorff capacity of*** E ***of dimension*** d, $0 < d \leq n$ *to be the following quantity*

$$H^d_\infty(E) = \inf \sum_j r^d_j$$

where the infimum is taken over all possible covers of E.

Remark 2.17 The name capacity may lead to confusion. Here we use this expression in the sense of N. Meyers. See [13], page 257.

Once we have this capacity, we can pass to the Choquet integral of $\phi \in C_0(\mathbb{R}^n)^+$:

Definition 2.18 (Choquet integral and $\mathbf{L^1(H^d_\infty)}$) *Let* $\phi \in C_0(\mathbb{R}^n)^+$. *Then the* ***Choquet integral*** *of* ϕ *with respect to the Hausdorff capacity* H^d_∞ *is defined to be the following Riemann integral:*

$$\int \phi \, dH^d_\infty \equiv \int_0^\infty H^d_\infty[\phi > \lambda] \, d\lambda.$$

The space $\mathbf{L^1(H^d_\infty)}$ *is now the completion of* $C_0(\mathbb{R}^n)$ *under the functional* $\int |\phi| \, dH^d_\infty$.

Two important facts about $L^1(H^d_\infty)$ are summarised below, again for instance see [1] and also the references given there.

Remark 2.19

- $L^1(H^d_\infty)$ can also be characterised to be the space of all H^d_∞-quasi continuous functions ϕ which satisfy $\int |\phi| \, dH^d_\infty < \infty$, i.e. for all $\varepsilon > 0$ there exists an open set G such that $H^d_\infty[G] < \varepsilon$ and that ϕ restricted to the complement of G is continuous there.

- One can show that $L^1(H^d_\infty)$ is a quasi-Banach space with respect to the quasi-norm $\int |\phi| \, dH^d_\infty$.

Now, we can state the duality result we mentioned earlier. A proof of this assertion is given in [1], but take care of the notation which differs from our notation!

Proposition 2.20 *We have $(L^1(H^d_\infty))^* = \mathcal{M}^{\frac{n}{n-d}}$ and in particular the estimate*

$$\Big| \int u \, d\mu \Big| \leq ||u||_{L^1(H^d_\infty)} ||\mu||_{\mathcal{M}^{\frac{n}{n-d}}}$$

holds and

$$||\mu||_{(L^1(H^d_\infty))^*} = \sup_{||u||_{L^1(H^d_\infty)} \leq 1} \Big| \int u \, d\mu \Big| \simeq ||\mu||_{\mathcal{M}^{\frac{n}{n-d}}}.$$

Note that in order to show that a certain function belongs to $\mathcal{M}^{\frac{n}{n-d}}$, it is enough to show that it defines a linear functional on $L^1(H^d_\infty)$, i. e. that $\sup_{||u||_{L^1(H^d_\infty)} \leq 1} \Big| \int u \, d\mu \Big| < \infty$. This does not require that $L^1(H^d_\infty)$ is a Banach space and is quite different from the case when you use the dual characterisation of a norm in order to show that a certain distribution belongs to a certain space.

Remark 2.21 The above proposition is just a special case of a more general result which involves also spaces $L^p(H^d_\infty)$, see for instance [2].

Before ending this section we will state some useful remarks for later applications.

Remark 2.22

- Observe that $\mathcal{M}^p \subset \mathcal{S}'$ (in particular for $p = \frac{n}{n-d}$). In order to verify this, note that $\mathcal{M}^p \subset N^0_{p,1,\infty} \subset \mathcal{S}'$: Let $\mu \in \mathcal{M}^p$ and let as usual $\varphi \in \Phi(\mathbb{R}^n)$ then we have

$$\begin{aligned}
||\mu|N^0_{p,1,\infty}|| &= \sup_{k\in\mathbb{N}} ||\check{\varphi}_k * \mu|M^p_1|| \\
&= \sup_{k\in\mathbb{N}} ||\check{\varphi}_k * \mu|M^p|| \\
&\quad \text{note that } \check{\varphi}_k * \mu \in C^\infty \subset L^1_{loc} \text{ since } \mu \in \mathcal{D}' \\
&\quad \text{and } \check{\varphi}_k * \mu \text{ can be seen as a measure} \\
&\leq \sup_{k\in\mathcal{N}} ||\check{\varphi}_k||_1 ||\mu|M^p|| \\
&\quad \text{because of [10], lemma 1.8} \\
&\leq C||\mu|\mathcal{M}^p|| \\
&< \infty \\
&\quad \text{according to our hypothesis.}
\end{aligned}$$

  Once we have this, we apply the continuous embedding of $N^0_{p,1,\infty}$ into $\mathcal{S}'$ (see e.g. [12]) and conclude that actually $\mathcal{M}^p \subset \mathcal{S}'$.
  Note also that $\mathcal{S} \subset L^1(H^d_\infty)$

- Using the duality asserted above, we can show that $L^1(H^d_\infty) \subset \mathcal{S}'$: We start with $f \in C_0^\infty(\mathbb{R}^n)$. Since $f \in L^\infty$ it is easy to check that $f \in M^p_q$, $1 \leq q \leq p < \infty$, with $||f|M^p_q|| = ||f||_\infty$. Moreover, f even belongs to $\mathcal{M}^p_q$. In order to establish this, it remains to show that there is a constant C, independent on f, such that $\forall x \in \mathbb{R}^n$ and for $1 \leq r$
$$||f||_{L^1(B_r(x))} \leq C r^{\frac{n}{q}-\frac{n}{p}}.$$
In fact, it holds $\forall x \in \mathbb{R}^n$ and $\forall r \geq 1$
$$\begin{aligned} ||f||_{L^1(B_r(x))} &\leq ||f||_1 \\ &\leq ||f||_1 r^{\frac{n}{q}-\frac{n}{p}} \\ &\quad \text{since due to the choice of } p \text{ and } q \text{ we have} \\ &\quad \frac{n}{q} - \frac{n}{p} \geq 0. \end{aligned}$$
If we put together all these information we find
$$||f|\mathcal{M}^p_q|| \leq ||f||_\infty + ||f||_1.$$
Now, recall that the duality between $L^1(H^d_\infty)$ and $\mathcal{M}^{\frac{n}{n-d}}$ is given by
$$< \mu, u >_{(L^1(H^d_\infty))^* = \mathcal{M}^{\frac{n}{n-d}}, L^1(H^d_\infty)} = \int u \, d\mu$$
where $u \in L^1(H^d_\infty)$ and $\mu \in \mathcal{M}^{\frac{n}{n-d}}$.
In a next step we define the action of $u \in L^1(H_\infty)$ on $f \in C_0^\infty$ as follows
$$< u, f >_{\mathcal{D}', C_0^\infty} := < f, u >_{\mathcal{M}^{\frac{n}{n-d}}, L^1(H^d_\infty)}.$$
Last, but not least, we observe that for $\varphi \in \mathcal{S}$ we have
$$||\varphi||_\infty + ||\varphi||_1 \leq C(n)||\varphi||_{\mathcal{S}}.$$
This finally leads to the conclusion that in fact, $L^1(H^d_\infty) \subset \mathcal{S}'$.

This last remark enables us to use the above introduced $L^1(H^d_\infty)$-quasi norm to construct - in analogy to the case of Besov- or Besov-Morrey-spaces - a new space of functions.

Definition 2.23 (Besov-Choquet spaces) *Let $\varphi \in \Phi(\mathbb{R}^n)$.
We say that $f \in \mathcal{S}'$ belongs to $\mathbf{B^0_{L^1(H^d_\infty),\infty}}$ if $\exists \{f_k(x)\}_{k=0}^\infty \subset L^1(H^d_\infty)$ such that the following holds*
$$f = \sum_{k=0}^\infty \mathcal{F}^{-1}\varphi_k \mathcal{F} f_k \text{ in } \mathcal{S}'(\mathbb{R}^n)$$
and
$$\sup_k ||f_k|L^1(H^d_\infty)|| < \infty.$$
Moreover we set
$$||f|B^0_{L^1(H^d_\infty),\infty}|| = \inf \sup_k ||f_k|L^1(H^d_\infty)||$$
*where the infimum is taken over all admissible representations of f.
Moreover, we denote by $\mathbf{b^0_{L^1(H^d_\infty),\infty}}$ the closure of $\mathcal{S}$ under the construction explained above.*

Remark 2.24 In complete analogy to the construction of the Besov spaces (respectively the Besov-Morrey-spaces) one could also construct new spaces if we replace the Lebesgue L^p-norms (respectively the Morrey-norms) by $L^p(H^d_\infty)$-quasi-norms.

3 Proofs

3.1 Some preliminary remarks

In what follows we set

$$\mathbf{f^j(x)} = \mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(x)$$

where $\varphi = \{\varphi_j(x)\}_{j=0}^{\infty} \in \Phi(\mathbb{R}^n)$.

Recall that once we can control the **paraproducts** $\pi_1(f,g) = \sum_{k=2}^{\infty} \sum_{l=0}^{k-2} f^l g^k$, $\pi_2(f,g) = \sum_{k=0}^{\infty} \sum_{l=k-1}^{k+1} f^l g^k$ and $\pi_3(f,g) = \sum_{l=2}^{\infty} \sum_{k=0}^{l-2} f^l g^k$ ($f^i = 0$ if $i \leq -1$ and similarly for g) we are also able to control the product fg (see e.g. [17]).

Since in the sequel we want to take into account cancellation phenomena, we will analysis

$$\pi_1(a_x, b_y)\ ,\ \pi_1(a_y, b_x)\ , \pi_3(a_x, b_y)\ , \pi_3(a_y, b_x) \text{ and } \sum_{s=0}^{\infty} \sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s. \quad (6)$$

Last but not least, remember that

$$supp\ \mathcal{F}\Big(\sum_{i=0}^{l-2} a_x^i b_y^l\Big) \subset \Big\{\xi \mid 2^{l-3} \leq |\xi| \leq 2^{l+3}\Big\} \text{ for } l \geq 2.$$

and

$$supp\ \mathcal{F}\Big(\sum_{i=l-1}^{l+1} a_x^i b_y^l\Big) \subset \Big\{\xi \mid |\xi| \leq 5 \cdot 2^l\Big\} \text{ for } l \geq 0.$$

3.2 Proof of theorem 1.2 i)

The proof of this assertion is split into several parts: In a first step we show that $\pi_1(a_x, b_y)$, $\pi_3(a_x, b_y)$, $\pi_3(a_y, b_x)$ and $\pi_1(a_y, b_x) \in B^{-1}_{\infty,1}$ and $\sum_{s=0}^{\infty} \sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s \in B^{-2}_{\infty,1}$. Once we have this we show in a second step that under this hypothesis the solution u of

$$-\Delta u = f \text{ where} f \in B^{-2}_{\infty,1}$$

is continuous.

Claim: $\pi_1(a_x, b_y) \in B^{-2}_{\infty,1}$

Our hypotheses together with [10], theorem 2.5, ensures us that $a_x,\ b_y \in B^{-1}_{\infty,2}$. Next, due to [17], proposition 1, chapter 2.3.2, it is enough to prove that

$$||2^{-2j} c_j | l^1(L^\infty)|| < \infty$$

where as before $c_j := \sum_{t=0}^{k-2} a_x^t b_y^j$.
We actually have

$$\begin{aligned}
||2^{-2j}c_j|l^1(L^\infty)|| &= \sum_{j=0}^{\infty} 2^{-2j}||\sum_{t=0}^{j-2} a_x^t b_y^j||_\infty \\
&\leq \sum_{j=0}^{\infty} 2^{-2j}||\sum_{t=0}^{j-2} a_x^t||_\infty ||b_y^j||_\infty \\
&= \sum_{j=0}^{\infty} 2^{-j}||\sum_{t=0}^{j-2} a_x^t||_\infty 2^{-j}||b_y^j||_\infty \\
&\leq \Big(\sum_{j=0}^{\infty} 2^{-2j}||\sum_{t=0}^{j-2} a_x^t||_\infty^2\Big)^{\frac{1}{2}} \Big(\sum_{j=0}^{\infty} 2^{-2j}||b_y^j||_\infty^2\Big)^{\frac{1}{2}} \\
&\quad \text{due to Hölder's inequality} \\
&= ||2^{-j}|\sum_{t=0}^{j-2} a_x^t|\ |l^2(L^\infty)||||b_y|B_\infty^{-1}|| \\
&\leq C||2^{-j}|\sum_{t=0}^{j} a_x^t|\ |l^2(L^\infty)||||b_y|B_\infty^{-1}|| \\
&\leq C||a_x|B_{\infty,2}^{-1}||||b_y|B_\infty^{-1}|| \text{ because of [17], first lemma in chapter 4.4.2} \\
&< \infty \text{ thanks to our hypothesis.}
\end{aligned}$$

This shows that in fact $\pi_1(a_x, b_y) \in B_{\infty,1}^{-2}$. Similarly one proves that also $\pi_1(a_y, b_x)$, $\pi_3(a_x, b_y)$ and $\pi_1(a_y, b_x)$ belong to the same space.
In remains to analyse the contribution where the frequencies are comparable. This is our next goal.

Analysis of $\sum_{s=0}^{\infty}\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s$

In stead of first applying the embedding result of Kozono/Yamazaki which embeds Morrey-Besov spaces into Besov spaces and then analysing a certain quantity, we invert the order of these steps in order to estimate $\sum_{s=0}^{\infty}\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s$.
We will use the following result concerning predual spaces of Morrey spaces .

Proposition 3.1 *The dual space of* $b^0_{L^1(H_\infty^{n-2}),\infty}$ *is the space* $B^0_{\mathcal{M}_1^{\frac{n}{2}},1}$.

Remark 3.2 The above result has the same flavour as (see for instance [17])

$$(b^0_{\infty,\infty})^* = B^0_{1,1}$$

Proof of proposition 3.1:
We have to show the two inclusion relations.
We start with $(b^0_{L^1(H_\infty^{n-2}),\infty})^* \supset B^0_{\mathcal{M}_1^{\frac{n}{2}},1}$:
Assume that $f \in B^0_{\mathcal{M}_1^{\frac{n}{2}},1} \subset N^0_{\frac{n}{2},1,1} \subset \mathcal{S}'$ and assume that $\psi \in b^0_{L^1(H_\infty^{n-2}),\infty}$. By

density we may assume that $\psi \in \mathcal{S}$. We have to show that $f \in (b^0_{L^1(H^{n-2}_\infty),\infty})^*$. To this end let $\sum_{k=0}^\infty \check{\varphi}_k * \psi_k$ be a representation of ψ with

$$\sup_k ||\psi_k||_{L^1(H^{n-2}_\infty)} \leq 2||\psi|b^0_{L^1(H^{n-2}_\infty),\infty}||.$$

Note that in our case - as a tempered distribution - f acts on ψ and we estimate

$$\begin{aligned}
|f(\psi)| &= |f(\sum_{k\geq 0} \check{\varphi}_k * \psi_k)| \\
&= \Big| f\Big(\sum_{k=o}^\infty \mathcal{F}^{-1}(\varphi_k \mathcal{F}\psi_k)\Big)\Big| \\
&= \Big|\sum_{k=0}^\infty f\Big(\mathcal{F}^{-1}(\varphi_k \mathcal{F}\psi_k)\Big)\Big| = \Big|\sum_{k=0}^\infty \int f\mathcal{F}^{-1}(\varphi_k \mathcal{F}\psi_k\Big)\Big| \\
&= \Big|\sum_{k=0}^\infty \psi_k \mathcal{F}(\varphi_k \mathcal{F}^{-1} f)\Big| = \Big|\sum_{k=0}^\infty \int \psi_k \; df\Big| \\
&\quad \text{where } df = \mathcal{F}(\varphi_k \mathcal{F}^{-1} f)\; d\lambda \text{ with } \lambda \text{ the Lebesgue measure} \\
&\leq \sum_{k=0}^\infty |\psi_k \mathcal{F}(\varphi_k \mathcal{F}^{-1} f)| \\
&\leq \sup_{k\geq 0} ||\psi_k||_{L^1(H^{n-2}_\infty)} \sum_{k=o}^\infty ||\mathcal{F}(\varphi_k \mathcal{F}^{-1} f)||_{\mathcal{M}^{\frac{n}{2}}} \text{ recall proposition 2.20} \\
&= \sup_{k\geq 0} ||\psi_k||_{L^1(H^{n-2}_\infty)} \sum_{k=o}^\infty ||\mathcal{F}(\varphi_k \mathcal{F}^{-1} f)||_{\mathcal{M}_1^{\frac{n}{2}}} \\
&\quad \text{cf. also remark 2.22} \\
&\leq C \sup_{k\geq 0} ||\psi_k||_{L^1(H^{n-2}_\infty)} \sum_{k=o}^\infty ||\mathcal{F}^{-1}(\varphi_k \mathcal{F} f)||_{\mathcal{M}_1^{\frac{n}{2}}} \\
&\leq C||\psi|b^0_{L^1(H^{n-2}_\infty),\infty}|| \;\; ||f|B^0_{\mathcal{M}_1^{\frac{n}{2}},1}|| \\
&< \infty \\
&\quad \text{thanks to our assumptions.}
\end{aligned}$$

Now we show the other inclusion, $(b^0_{L^1(H^{n-2}_\infty),\infty})^* \subset B^0_{\mathcal{M}_1^{\frac{n}{2}},1}$:
We start with $f \in (b^0_{L^1(H^{n-2}_\infty),\infty})^*$ and we have to show that f belongs also to $B^0_{\mathcal{M}_1^{\frac{n}{2}},1}$: First of all, note that f gives also rise to elements of $(L^1(H^{n-2}_\infty))^*$ as follows: Each $\psi \in b^0_{L^1(H^{n-2}_\infty),\infty}$ can be seen as a sequence $\{\psi_k\}_{k=0}^\infty \subset L^1(H^{n-2}_\infty)$, and of course $\check{\varphi}_k * \psi_k \in b^0_{L^1(H^{n-2}_\infty),\infty}$ $\forall k \in \mathbb{N}$. Moreover, for each $k \in \mathbb{N}$ we have

- again by density of $\mathcal{S}$ -

$$\begin{aligned} f(\delta_{kj}(\check{\varphi}_j * \psi_j)) &= < f, \delta_{kj}\psi >_{(b^0_{L^1(H^{n-2}_\infty),\infty})^*, b^0_{L^1(H^{n-2}_\infty),\infty}} \\ &= < f, \check{\varphi}_k * \psi_k >_{(b^0_{L^1(H^{n-2}_\infty),\infty})^*, b^0_{L^1(H^{n-2}_\infty),\infty}} \\ &= < f, \check{\varphi}_k * \psi_k >_{\mathcal{S}',\mathcal{S}} \\ &= < f, \mathcal{F}^{-1}(\varphi_k \mathcal{F}\psi_k) >_{\mathcal{S}',\mathcal{S}} \\ &= < \mathcal{F}(\varphi_k \mathcal{F}^{-1} f), \psi_k >_{\mathcal{S}',\mathcal{S}} \\ &= < \mathcal{F}(\varphi_k \mathcal{F}^{-1} f), \psi_k >_{\mathcal{M}^{\frac{n}{2}}, L^1(H^{n-2}_\infty)} . \end{aligned}$$

Next we will construct a special element of $b^0_{L^1(H^{n-2}_\infty),\infty}$:
Let $0 < \varepsilon$ small.
We choose ψ_k such that

- $\psi_k \in \mathcal{S}$: Remember that we have density!
- $||\psi_k||_{L^1(H^{n-2}_\infty)} \leq 1$
- $0 < \ < \mathcal{F}(\varphi_k \mathcal{F}^{-1} f), \psi_k >_{\mathcal{M}^{\frac{n}{2}}, L^1(H^{n-2}_\infty)}$
- $$\begin{aligned} < \mathcal{F}(\varphi_k \mathcal{F}^{-1} f), \psi_k >_{\mathcal{M}^{\frac{n}{2}}, L^1(H^{n-2}_\infty)} &\geq ||\mathcal{F}(\varphi_k \mathcal{F}^{-1} f)||_{\mathcal{M}^{\frac{n}{2}}} - \varepsilon 2^{-k} \\ &= ||\mathcal{F}(\varphi_k \mathcal{F}^{-1} f)||_{(L^1(H^{n-2}_\infty))^*} - \varepsilon 2^{-k} \\ &= \sup_{\substack{u \in L^1(H^{n-2}_\infty) \\ ||u||_{L^1(H^{n-2}_\infty)} \leq 1}} | < \mathcal{F}(\varphi_k \mathcal{F}^{-1} f), u > | - \varepsilon 2^{-k}. \end{aligned}$$

Note that like that $\psi = \sum_{k=0}^\infty \check{\varphi}_k * \psi_k \in b^0_{L^1(H^{n-2}_\infty),\infty}$ with

$$||\psi | b^0_{L^1(H^{n-2}_\infty),\infty}|| \leq 1.$$

If we put now all this together we find - recall that f acts linearly! -

$$\begin{aligned} \sum_{k=0}^\infty ||f^k||_{\mathcal{M}_1^{\frac{n}{2}}} &= \sum_{k=0}^\infty ||\mathcal{F}^{-1}(\varphi_k \mathcal{F} f)||_{\mathcal{M}_1^{\frac{n}{2}}} \\ &= C \sum_{k=0}^\infty ||\mathcal{F}(\varphi_k \mathcal{F}^{-1} f)||_{\mathcal{M}_1^{\frac{n}{2}}} \\ &\leq 2\varepsilon + f(\psi) \\ & \quad \psi \text{ as constructed above} \\ &\leq 2\varepsilon + ||f|(b^0_{L^1(H^{n-2}_\infty),\infty})^*|| \ \ ||\psi|b^0_{L^1(H^{n-2}_\infty),\infty}|| \\ &\leq 2\varepsilon + ||f|(b^0_{L^1(H^{n-2}_\infty),\infty})^*||. \end{aligned}$$

Since this holds for all $0 < \varepsilon$ we let ε tend to zero and get the desired inclusion.
All together we established the duality result we claimed above.

□

What concerns the next lemma, recall that $\mathcal{S}$ is dense in $b^0_{L^1(H^{n-2}_\infty),\infty}$:

Lemma 3.3 *Let $\phi \in \Phi(\mathbb{R}^n)$ and assume that $\psi \in \mathcal{S} \cap L^1(H^{n-2}_\infty)$ with representation $\{\psi_k\}_{k=0}^\infty$, i.e. $\sum_{k=0}^\infty \check{\varphi}_k * \psi_k = \psi$, such that*

$$\sup_k ||\psi_k||_{L^1(H^{n-2}_\infty)} \leq 2||\psi|b^0_{L^1(H^{n-2}_\infty),\infty}||.$$

Then

$$\begin{aligned}
\Big|\Big|\frac{\partial}{\partial x}\check{\varphi}_k * \psi_k\Big|\Big|_{L^1(H^{n-2}_\infty)} &= \Big|\Big|\frac{\partial}{\partial x}(\check{\varphi}_k * \psi_k)\Big|\Big|_{L^1(H^{n-2}_\infty)} \\
&\leq C2^s||\psi_k||_{L^1(H^{n-2}_\infty)} \leq C2^s||\psi|b^0_{L^1(H^{n-2}_\infty),\infty}||.
\end{aligned}$$

Proof:
For the proof of this lemma, we need the fact that if $f(x) \geq 0$ is lower semi-continuous on $\mathbb{R}^n$ then

$$||f||_{L^1(H^d_\infty)} = \int f \, dH^d_\infty \sim \sup\left\{\int f \, d\mu | \mu \in \mathcal{M}_+^{\frac{n}{n-d}} \text{ and } ||\mu||_{\mathcal{M}^{\frac{n}{n-d}}} \leq 1\right\}.$$

see Adams [1].

It holds

$$\begin{aligned}
\Big|\Big|\frac{\partial}{\partial x}\check{\varphi}_k * \psi_k\Big|\Big|_{L^1(H^{n-2}_\infty)} &\leq || \, |\frac{\partial}{\partial x}\check{\varphi}_k| * |\psi_k| \, ||_{L^1(H^{n-2}_\infty)} \\
&\leq C \sup_{\substack{\mu \in \mathcal{M}_+^{\frac{n}{2}} \\ ||\mu||_{\mathcal{M}^{\frac{n}{2}}} \leq 1}} \left\{\int |\frac{\partial}{\partial x}\check{\varphi}_k| * |\psi_k| \, d\mu\right\} \\
&= C \sup_{\substack{\mu \in \mathcal{M}_+^{\frac{n}{2}} \\ ||\mu||_{\mathcal{M}^{\frac{n}{2}}} \leq 1}} \left\{\int \int |\frac{\partial}{\partial x}\check{\varphi}_k|(x-y)|\psi_k|(y) \, d\lambda(y) d\mu(x)\right\} \\
&\leq C \sup_{\substack{\mu \in \mathcal{M}_+^{\frac{n}{2}} \\ ||\mu||_{\mathcal{M}^{\frac{n}{2}}} \leq 1}} \left\{\int |\psi_k|(y) \int |\frac{\partial}{\partial x}\check{\varphi}_k|(x-y) \, d\mu(x) d\lambda(y)\right\} \\
&\quad \text{by Tonelli's theorem} \\
&= C \sup_{\substack{\mu \in \mathcal{M}_+^{\frac{n}{2}} \\ ||\mu||_{\mathcal{M}^{\frac{n}{2}}} \leq 1}} \left\{\int |\psi_k|(y) \int |\frac{\partial}{\partial x}\check{\varphi}_k|(y-x) \, d\mu(x) d\lambda(y)\right\} \\
&\quad \text{note that } \varphi_k \text{ can be chose radial} \\
&\quad \text{which implies that } \check{\varphi}_k \text{ and } \frac{\partial}{\partial x}\check{\varphi}_k \text{ are radial} \\
&\quad \text{see e.g. [22]} \\
&= C \sup_{\substack{\mu \in \mathcal{M}_+^{\frac{n}{2}} \\ ||\mu||_{\mathcal{M}^{\frac{n}{2}}} \leq 1}} \left\{\int |\psi_k|(y) \frac{\partial}{\partial x}\check{\varphi}_k|(y-x) * \mu(y) \, d\lambda(y)\right\}
\end{aligned}$$

and we continue

$$
\begin{aligned}
\Big|\Big|\frac{\partial}{\partial x}\check{\varphi}_k * \psi_k\Big|\Big|_{L^1(H_\infty^{n-2})} &\leq C \sup_{\substack{\mu\in\mathcal{M}_+^{\frac{n}{2}} \\ ||\mu||_{\mathcal{M}^{\frac{n}{2}}}\leq 1}} \left\{\int |\psi_k|(y)\, d\nu(y)\right\} \\
&\quad \text{where } \nu := \frac{\partial}{\partial x}\check{\varphi}_k\lambda * \mu \\
&\leq C \sup_{\substack{\mu\in\mathcal{M}_+^{\frac{n}{2}} \\ ||\mu||_{\mathcal{M}^{\frac{n}{2}}}\leq 1}} \left\{||\psi_k||_{L^1(H_\infty^{n-2})}\, ||\frac{\partial}{\partial x}\check{\varphi}_k\lambda * \mu||_{\mathcal{M}^{\frac{n}{2}}}\right\} \\
&\leq C \sup_{\substack{\mu\in\mathcal{M}_+^{\frac{n}{2}} \\ ||\mu||_{\mathcal{M}^{\frac{n}{2}}}\leq 1}} \left\{||\psi_k||_{L^1(H_\infty^{n-2})}\, ||\frac{\partial}{\partial x}\check{\varphi}_k||_{L^1}\, ||\mu||_{\mathcal{M}^{\frac{n}{2}}}\right\} \\
&\quad \text{by [10], lemma 1.8} \\
&\leq C||\psi_k||_{L^1(H_\infty^{n-2})}\, ||\frac{\partial}{\partial x}\check{\varphi}_k||_{L^1} \\
&\leq C2^k||\psi_k||_{L^1(H_\infty^{n-2})} \\
&\leq C2^k||\psi|b^0_{L^1(H_\infty^{n-2}),\infty}||
\end{aligned}
$$

what we had to prove.

$\square$

The next lemma is a technical one:

Lemma 3.4 *Let a and b belong to $C_0^\infty(\mathbb{R}^n)$, $t = s + j$ where $j \in \{-1, 0, 1\}$ and ψ with representation $\{\psi_k\}_{k=0}^\infty$, i.e. $\sum_{k=0}^\infty \check{\varphi}_k * \psi_k = \psi$, such that*

$$\sup_k ||\psi_k||_{L^1(H_\infty^{n-2})} \leq 2||\psi|b^0_{L^1(H_\infty^{n-2}),\infty}|| \leq 2$$

Then

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \frac{\partial}{\partial x}(a^t b^s_y)\psi - \frac{\partial}{\partial y}(a^t b^s_x)\psi \\
&= \int_{\mathbb{R}^n} \frac{\partial}{\partial x}\Big(a^t b^s_y\Big)\Big(\sum_{k=0}^{s+3}\mathcal{F}^{-1}(\varphi_k\mathcal{F}\psi_k)\Big) - \frac{\partial}{\partial y}\Big(a^t b^s_x\Big)\Big(\sum_{k=0}^{s+3}\mathcal{F}^{-1}(\varphi_k\mathcal{F}\psi_k)\Big).
\end{aligned}
$$

Proof:
First of all, note that $h \in \mathcal{S}'$ and $a^t b^s_y$ and $a^t b^s_x$ belong to $\mathcal{S}$ independently of the choices of s and t.
We now calculate

$$
\begin{aligned}
\int_{\mathbb{R}^n} \frac{\partial}{\partial x}(a^t b^s_y)\psi - \frac{\partial}{\partial y}(a^t b^s_x)\psi &= \int_{\mathbb{R}^n} \frac{\partial}{\partial x}(a^t b^s_y)\psi - \int_{\mathbb{R}^n} \frac{\partial}{\partial y}(a^t b^s_x)\psi \\
&= \int_{\mathbb{R}^n} \frac{\partial}{\partial x}(a^t b^s_y)\sum_{k=0}^{\infty}\mathcal{F}^{-1}(\varphi_k\mathcal{F}\psi_k) \\
&\quad - \int_{\mathbb{R}^n} \frac{\partial}{\partial y}(a^t b^s_x)\sum_{k=0}^{\infty}\mathcal{F}^{-1}(\varphi_k\mathcal{F}\psi_k)
\end{aligned}
$$

so

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{\partial}{\partial x}(a^t b_y^s)\psi - \frac{\partial}{\partial y}(a^t b_x^s)\psi \quad &= \quad \int_{\mathbb{R}^n} \frac{\partial}{\partial x}(a^t b_y^s)\Big[\sum_{k=0}^{s+3} \mathcal{F}^{-1}(\varphi_k \mathcal{F}\psi_k) + \sum_{k=s+4}^{\infty} \mathcal{F}^{-1}(\varphi_k \mathcal{F}\psi_k)\Big] \\ &\quad - \int_{\mathbb{R}^n} \frac{\partial}{\partial y}(a^t b_x^s)\Big[\sum_{k=0}^{s+4} \mathcal{F}^{-1}(\varphi_k \mathcal{F}\psi_k) + \sum_{k=s+4}^{\infty} \mathcal{F}^{-1}(\varphi_k \mathcal{F}\psi_k)\Big].\end{aligned}$$

These calculations show that we have to prove that

$$\int_{\mathbb{R}^n} \frac{\partial}{\partial x}(a^t b_y^s) \sum_{k=s+4}^{\infty} \mathcal{F}^{-1}(\varphi_k \mathcal{F}\psi_k) = 0$$

and

$$\int_{\mathbb{R}^n} \frac{\partial}{\partial y}(a^t b_x^s) \sum_{k=s+4}^{\infty} \mathcal{F}^{-1}(\varphi_k \mathcal{F}\psi_k) = 0.$$

In what follows, we will only discuss the first integral because the second one can be analysed in exactly the same way.
So from now on we look at

$$\int_{\mathbb{R}^n} \frac{\partial}{\partial x}(a^t b_y^s) \sum_{k=s+4}^{\infty} \mathcal{F}^{-1}(\varphi_k \mathcal{F}\psi_k).$$

Here we have

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{\partial}{\partial x}(a^t b_y^s) \sum_{k=s+4}^{\infty} \mathcal{F}^{-1}(\varphi_k \mathcal{F}\psi_k) \quad &= \quad \int_{\mathbb{R}^n} \frac{\partial}{\partial x}(a^t b_y^s)\mathcal{F}^{-1}\Big(\sum_{k=s+4}^{\infty} \varphi_k \mathcal{F}\psi_k\Big) \\ &\quad \text{sine the sum is locally finite} \\ &= \quad \int_{\mathbb{R}^n} \frac{\partial}{\partial x}(a^t b_y^s)\mathcal{F}\mathcal{F}^{-1}\mathcal{F}^{-1}\Big(\sum_{k=s+4}^{\infty} \varphi_k \mathcal{F}\psi_k\Big) \\ &= \quad (2\pi)^n \int_{\mathbb{R}^n} \mathcal{F}\Big(\frac{\partial}{\partial_x}(a^t b_y^s)\Big) \sum_{k=s+4}^{\infty} \varphi_k(-\,\cdot)\mathcal{F}\psi_k(-\,\cdot) \\ &\quad \text{because } \frac{\partial}{\partial x}(a^t b_y^s) \in \mathcal{S} \text{ and } \sum_{k=s+4}^{\infty} \varphi_k \mathcal{F}\psi_k) \in \mathcal{S}' \\ &= \quad 0.\end{aligned}$$

In the last step of the above calculations we used the fact that

$$supp\ \mathcal{F}(\frac{\partial}{\partial x}(a^t b_y^s)) \subset \{\xi|\ |\xi| \leq 5 \cdot 2^s\}$$

and

$$supp \sum_{k=s+4}^{\infty} \varphi_k(-\,\cdot) \subset \big\{\xi|\ 2^{s+3} \leq |\xi|\big\}$$

imply that

$$supp\ \mathcal{F}\Big(\frac{\partial}{\partial x}(a^t b_y^s)\Big) \cap supp \sum_{k=s+4}^{\infty} \varphi_k = \emptyset.$$

This completes the proof.

□

Now, we can start with the estimate of $\sum_{s=0}^{\infty}\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s$. Our goal is to show that $\sum_{s=0}^{\infty}\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s$ belongs to $B^0_{\mathcal{M}_1^{\frac{n}{2}},1}$. Making use of the above duality result, see proposition 3.1, we will first show that

$$\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s \in B^0_{\mathcal{M}_1^{\frac{n}{2}},1} \quad \forall s \in \mathbb{N}$$

then we establish

$$\sum_{s=0}^{\infty}\Big|\Big|\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s \Big| B^0_{\mathcal{M}_1^{\frac{n}{2}},1}\Big|\Big| < \infty.$$

This ensures that

$$\sum_{s=0}^{\infty}\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s \in B^0_{\mathcal{M}_1^{\frac{n}{2}},1} \subset N^0_{\frac{n}{2},1,1}.$$

First of all, let us fix $t = s + j$ where $j \in \{-1, 0, 1\}$.
In order to show that $a_x^t b_y^s - a_y^t b_x^s \in B^0_{\mathcal{M}_1^{\frac{n}{2}},1}$ it suffices to show that for all $\psi \in b^0_{L^1(H^{n-2}_\infty),\infty}$ with $||\psi|b^0_{L^1(H^{n-2}_\infty),\infty}|| \le 1$ the following inequality holds

$$\int_{\mathbb{R}^n} \psi \; d(a_x^t b_y^s - a_y^t b_x^s) = \int_{\mathbb{R}^n} \psi(a_x^t b_y^s - a_y^t b_x^s)\; d\lambda < \infty$$

where as before λ denotes the Lebesgue measure.
Moreover, in the subsequent calculations we assume that for ψ we have a representation $\{\psi_k\}_{k=0}^{\infty}$, i.e. $\sum_{k=0}^{\infty} \check{\varphi}_k * \psi_k = \psi$, such that

$$\sup_k ||\psi_k||_{L^1(H^{n-2}_\infty)} \le 2||\psi|b^0_{L^1(H^{n-2}_\infty),\infty}|| \le 2$$

and again, recall that we have density of $\mathcal{S}$ in $b^0_{L^1(H^{n-2}_\infty),\infty}$.
In this case we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \psi(a_x^t b_y^s - a_y^t b_x^s) &= \int_{\mathbb{R}^n} \psi \frac{\partial}{\partial x}\big(a^t b_y^s\big) - \psi \frac{\partial}{\partial y}\big(a^t b_x^s\big) \\
&= \int_{\mathbb{R}^n} \Big[\frac{\partial}{\partial x}\big(a^t b_y^s\big)\Big(\sum_{k=0}^{s+3} \mathcal{F}^{-1}(\varphi_k \mathcal{F}\psi_k)\Big) \\
&\quad -\frac{\partial}{\partial y}\big(a^t b_x^s\big)\Big(\sum_{k=0}^{s+3} \mathcal{F}^{-1}(\varphi_k \mathcal{F}\psi_k)\Big)\Big] \\
&\quad \text{because of the same reason as in lemma 3.4} \\
&= \int_{\mathbb{R}^n} \Big[- a^t b_y^s \frac{\partial}{\partial x}\Big(\sum_{k=0}^{s+3} \mathcal{F}^{-1}(\varphi_k \mathcal{F}\psi_k)\Big) \\
&\quad + a^t b_x^s \frac{\partial}{\partial y}\Big(\sum_{k=0}^{s+3} \mathcal{F}^{-1}(\varphi_k \mathcal{F}\psi_k)\Big)\Big] \\
&\quad \text{by a simple integration by parts}
\end{aligned}$$

and further

$$
\begin{aligned}
\int_{\mathbb{R}^n} \psi(a_x^t b_y^s - a_y^t b_x^s) &\le \int_{\mathbb{R}^n} \Big[- a^t b_y^s \Big(\sum_{k=0}^{s+3} \frac{\partial}{\partial x} \check{\varphi}_k * \psi_k \Big) \\
&\quad + a^t b_x^s \Big(\sum_{k=0}^{s+3} \frac{\partial}{\partial y} \check{\varphi}_k * \psi_k \Big) \Big] \\
&\le \sum_{k=0}^{s+3} \int_{\mathbb{R}^n} \Big[- a^t b_y^s \frac{\partial}{\partial x} \check{\varphi}_k * \psi_k \\
&\quad + a^t b_x^s \frac{\partial}{\partial y} \check{\varphi}_k * \psi_k \Big] \\
&\le \sum_{k=0}^{s+3} \Big(||a^t b_y^s | \mathcal{M}^{\frac{n}{2}}|| \, || \frac{\partial}{\partial x} \check{\varphi}_k * \psi_k | L^1(H_\infty^{n-2})|| \\
&\quad + ||a^t b_x^s | \mathcal{M}^{\frac{n}{2}}|| \, || \frac{\partial}{\partial y} \check{\varphi}_k * \psi_k | L^1(H_\infty^{n-2})|| \Big) \\
&\quad \text{by proposition 2.20} \\
&\le \sum_{k=0}^{s+3} \Big(||a^t b_y^s | \mathcal{M}_1^{\frac{n}{2}}|| \, || \frac{\partial}{\partial x} \check{\varphi}_k * \psi_k | L^1(H_\infty^{n-2})|| \\
&\quad + ||a^t b_x^s | \mathcal{M}_1^{\frac{n}{2}}|| \, || \frac{\partial}{\partial y} \check{\varphi}_k * \psi_k | L^1(H_\infty^{n-2})|| \Big) \\
&\le \sum_{k=0}^{s+3} \Big(||a^t | \mathcal{M}_2^n|| \, ||b_y^s | \mathcal{M}_2^n|| \, || \frac{\partial}{\partial x} \check{\varphi}_k * \psi_k | L^1(H_\infty^{n-2})|| \\
&\quad + ||a^t | \mathcal{M}_2^n|| \, ||b_x^s | \mathcal{M}_2^n|| \, || \frac{\partial}{\partial y} \check{\varphi}_k * \psi_k | L^1(H_\infty^{n-2})|| \Big) \\
&\quad \text{because of Hölder's inequality with Morrey norms} \\
&\quad \text{see also remark below} \\
&\le \sum_{k=0}^{s+3} \Big(||a^t | \mathcal{M}_2^n|| \, ||b_y^s | \mathcal{M}_2^n|| \, 2^k ||\psi | b^0_{L^1(H_\infty^{n-2}),\infty}|| \\
&\quad + ||a^t | \mathcal{M}_2^n|| \, ||b_x^s | \mathcal{M}_2^n|| \, 2^k ||\psi | b^0_{L^1(H_\infty^{n-2}),\infty}|| \Big) \\
&\quad \text{according to lemma 3.3} \\
&\le C 2^s ||a^t | \mathcal{M}_2^n|| \, ||b_y^s | \mathcal{M}_2^n|| + C 2^s ||a^t | \mathcal{M}_2^n|| \, ||b_x^s | \mathcal{M}_2^n|| \\
&< \infty \\
&\quad \text{due to our assumptions.}
\end{aligned}
$$

Thus we have seen that for all $s \in \mathbb{N}$

$$
a_x^t b_y^s - a_y^t b_x^s \in (b^0_{L^1(H_\infty^{n-2}),\infty})^* = B^0_{\mathcal{M}_1^{\frac{n}{2}},1} \subset N^0_{\frac{n}{2},1,1}.
$$

Next, we study

$$
\sum_{s=0}^{\infty} \Big|\Big| \sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s \Big| B^0_{\mathcal{M}_1^{\frac{n}{2}},1} \Big|\Big|.
$$

What concerns this latter quantity, we will assume for the sake of simplicity that $t = s$. Then we can estimate

$$\begin{aligned}
\sum_{s=0}^{\infty} ||a_x^s b_y^s - a_y^s b_x^s | B^0_{\mathcal{M}_1^{\frac{n}{2}},1}|| &= ||a_x^0 b_y^0 - a_y^0 b_x^0 | B^0_{\mathcal{M}_1^{\frac{n}{2}},1}|| + \sum_{s=1}^{\infty} ||a_x^s b_y^s - a_y^s b_x^s | B^0_{\mathcal{M}_1^{\frac{n}{2}},1}|| \\
&\le C||a^0|\mathcal{M}_2^n|| \, ||b_y^0|\mathcal{M}_2^n|| + C||a^0|\mathcal{M}_2^n|| \, ||b_x^0|\mathcal{M}_2^n|| \\
&\quad + C\sum_{s=1}^{\infty} 2^s ||a^s|\mathcal{M}_2^n|| \, ||b_y^s|\mathcal{M}_2^n|| \\
&\quad + C\sum_{s=1}^{\infty} 2^s ||a^s|\mathcal{M}_2^n|| \, ||b_x^s|\mathcal{M}_2^n|| \\
&\le C||a^0|\mathcal{M}_2^n|| \, ||b_y^0|\mathcal{M}_2^n|| + C||a^0|\mathcal{M}_2^n|| \, ||b_x^0|\mathcal{M}_2^n|| \\
&\quad + C\sum_{s=1}^{\infty} ||a_x^s|\mathcal{M}_2^n|| \, ||b_y^s|\mathcal{M}_2^n|| \\
&\quad + C\sum_{s=1}^{\infty} ||a_y^s|\mathcal{M}_2^n|| \, ||b_x^s|\mathcal{M}_2^n|| \\
&\quad \text{similar to } 2^{ms}||g||_p \simeq ||\nabla^m g||_p \text{ (under appropriate assumptions)} \\
&\quad \text{cf. also theorem 2.9 in [10]} \\
&\le C||a^0|\mathcal{M}_2^n|| \, ||b_y^0|\mathcal{M}_2^n|| + C||a^0|\mathcal{M}_2^n|| \, ||b_x^0|\mathcal{M}_2^n|| \\
&\quad + C\Big(\sum_{s=1}^{\infty} ||a_x^s|\mathcal{M}_2^n||^2\Big)^{\frac{1}{2}} \Big(\sum_{s=1}^{\infty} ||b_y^s|\mathcal{M}_2^n||^2\Big)^{\frac{1}{2}} \\
&\quad + C\Big(\sum_{s=1}^{\infty} ||a_y^s|\mathcal{M}_2^n||^2\Big)^{\frac{1}{2}} \Big(\sum_{s=1}^{\infty} ||b_x^s|\mathcal{M}_2^n||^2\Big)^{\frac{1}{2}} \\
&\quad \text{by Hölder's inequality} \\
&< \infty \\
&\quad \text{thanks to our hypothesis.}
\end{aligned}$$

All together we have seen that

$$\sum_{s=0}^{\infty} a_x^s b_y^s - a_y^s b_x^s \in B^0_{\mathcal{M}_1^{\frac{n}{2}},1} \subset N^0_{\frac{n}{2},1,1}.$$

Now, since the above estimate is independent of the choice of j we immediately conclude that

$$\sum_{s=0}^{\infty} \sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s \in N^0_{\frac{n}{2},1,1}$$

Now, as we know that $\sum_{s=0}^{\infty} \sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s \in B^0_{\mathcal{M}_1^{\frac{n}{2}},1} \subset N^0_{\frac{n}{2},1,1}$ we apply the embedding result of Kozono/Yamazaki, theorem 2.5 in [10], and find that

$$\sum_{s=0}^{\infty} \sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s \in B^{-2}_{\infty,1}.$$

Remark 3.5 Assume that $f, g \in \mathcal{M}_2^n$. Then we have for all $0 < r$ and for all $x \in \mathbb{R}^n$

$$\begin{aligned} ||fg||_{L^1(B_r(x))} &\leq ||f||_{L^2(B_r(x))}||g||_{L^2(B_r(x))} \\ &\leq C_1 r^{\frac{n}{2}-1} C_2 r^{\frac{n}{2}-1} \\ &= C r^{n-2}. \end{aligned}$$

According to the definition, this shows that $fg \in \mathcal{M}_1^{\frac{n}{2}}$.

Regularity

We rewrite our equation $\Delta u = f$ as $\Delta u = f^0 + \sum_{k \geq 1} f^k$.
And the solution u can be written as

$$\begin{aligned} u &= \Delta^{-1} f^0 + \Delta^{-1}(\sum_{k \geq 1} f^k) \\ &=: u_1 + u_2. \end{aligned}$$

Our strategy is to show that u_1 as well as u_2 is continuous and bounded.

What concerns u_1, observe that due to the Paley-Wiener theorem f^0 is analytic, so in particular continuous. This implies immediately - by classical results (see e.g. [8]) - that u_1 is continuous.
On one hand we have that $f^0 \in B^s_{\frac{n}{2},2}$ for all $s \in \mathbb{R}$ (since $\nabla a, \nabla b \in B^0_{\mathcal{M}_2^n,2} \subset \mathcal{M}_2^n \subset L^n$) on the other hand we know that $f^0 \in B^s_{\infty,1}$ for all $s \in \mathbb{R}$ because $f \in B^{-2}_{\infty,1}$. From that we can deduce by standard elliptic estimates (see also [17]) and the embedding result of Sickel and Triebel [21] that u_1 is not only continuous but also bounded!
Next, we will show that u_2 is bounded and continuous. In order to reach this goal, we show that $u_2 \in B^0_{\infty,1}$: We find the following estimates

$$\begin{aligned} ||u_2|B^0_{\infty,1}|| &= \sum_{s=0}^{\infty} ||u_2^s||_\infty \\ &= \sum_{s=0}^{\infty} 2^{-2s} 2^{2s} ||u_2^s||_\infty \\ &= C \sum_{s=0}^{\infty} 2^{-2s} ||(\Delta u_2)^s||_\infty \end{aligned}$$

This last passage holds thanks to the fact that

$$2^{ms}||g||_p \simeq ||\nabla^m g||_p$$

if the Fourier transform of g is supported on an annulus with radii comparable to 2^s(see [23] for instance).
For $s = 0$ we observe

$$\mathcal{F}(-\Delta u_2) = \mathcal{F}(\sum_{k \geq 1} f^k)$$

which implies

$$supp(\mathcal{F}(u_2)) \subset (B_1(0))^c$$

because of the fact that

$$supp(\mathcal{F}(\sum_{k\geq 1} f^k)) \subset (B_1(0))^c.$$

So in this case too, we can apply the above mentioned fact in order to conclude that also for $s = 0$ we have

$$||u_2^0||_\infty \leq C||(\Delta u_2)^0||_\infty.$$

Back to our estimate, we continue

$$\begin{aligned}
||u_2|B^0_{\infty,1}|| &\leq C\sum_{s=0}^{\infty} 2^{-2s}||(\Delta u_2)^s||_\infty \\
&= C\sum_{s=0}^{\infty} 2^{-2s}||(\sum_{k\geq 1} f^k)^s||_\infty \\
&= C\sum_{s=0}^{\infty} 2^{-2s}||\mathcal{F}^{-1}(\sum_{k=s-1}^{s+1} \varphi_s\varphi_k \hat{f})||_\infty \\
&\leq \sum_{s=0}^{\infty} 2^{-2s}||f^s||_\infty \\
&\quad \text{thanks to a Fourier multiplier result} \\
&\quad \text{for further details we refer to [25]} \\
&= C||f|B^{-2}_{\infty,1}|| \\
&< \infty \text{ according to our assumptions.}
\end{aligned}$$

This shows that u_2 belongs to $B^0_{\infty,1}(\mathbb{R}^n)$.
Alternatively one could make use of the lifting property, see [17], chapter 2.6, to show that $u_2 \in C$. (Recall that C denotes the space of all uniformly continuous functions on $\mathbb{R}^n$.) The last ingredient is the embedding result due to Sickel/Triebel (see [21]).
This leads immediately to the assertion we claimed because u as a sum of two bounded continuous functions is again continuous and bounded.

□

3.3 Proof of theorem 1.2 ii)

In a first step we show that $a_x b_y - a_y b_x \in B^{-1}_{\mathcal{M}_2^n,1}$:
From the proof of theorem 1.2 we know that

$$\sum_{k=0}^{\infty} \sum_{s=k-1}^{k+1} a_x^k b_y^s - a_y^k b_x^s \in B^0_{\mathcal{M}_1^{\frac{n}{2}},1} \subset B^{-1}_{\mathcal{M}_2^n,1}.$$

Next, we observe that

$$\begin{aligned}
||\pi_3(a_x, b_x)|B^{-1}_{\mathcal{M}^n_2,1}|| &\leq C\sum_{s=0}^{\infty} 2^{-s}\Big|\Big|\sum_{k=0}^{s-2} a^s_x b^k_y\Big|\Big|_{\mathcal{M}^n_2} \\
&\quad \text{by a simple modification of lemma 3.16 in [12]} \\
&\leq C\sum_{s=0}^{\infty} 2^{-s}||a^s_x||_{\mathcal{M}^n_2}\Big|\Big|\sum_{k=0}^{s-2} b^k_y\Big|\Big|_{\infty} \\
&\leq C\Big(\sum_{s=0}^{\infty}||a^s_x||^2_{\mathcal{M}^n_2}\Big)^{\frac{1}{2}}\Big(\sum_{s=0}^{\infty}2^{-2s}\Big|\Big|\sum_{k=0}^{s-2} b^k_y\Big|\Big|^2_{\infty}\Big)^{\frac{1}{2}} \\
&\leq C||a_x|B^0_{\mathcal{M}^n_2,2}||\Big(\sum_{s=0}^{\infty}2^{-2s}\Big|\Big|\sum_{k=0}^{s} b^k_y\Big|\Big|^2_{\infty}\Big)^{\frac{1}{2}} \\
&\leq C||a_x|B^0_{\mathcal{M}^n_2,2}||\ ||b_y|B^{-1}_{\mathcal{M}^n_2,2}|| \\
&\quad \text{according to lemma 4.4.2 of [17]} \\
&\leq ||a_x|B^0_{\mathcal{M}^n_2,2}||\ ||b_y|B^0_{\mathcal{M}^n_2,2}||.
\end{aligned}$$

Now, since

$$\partial_{x_i} u = \mathcal{F}^{-1}\Big(i\frac{\xi_i}{|\xi|^2}\mathcal{F}(\Delta u)\Big)$$

we note first, that due to the facts that $\Delta u \in F^0_{1,2} \subset L^1$ and $r^{-1} \in L^{\frac{n}{n-1}}$ for $n \geq 3$,

$$(\nabla u)^0 \in L^n \subset \mathcal{M}^n_2$$

which implies that $(\nabla u)^0 \in B^0_{\mathcal{M}^n_2,2}$.
Second, for $s \geq 1$ we have

$$||(\nabla u)^s||_{\mathcal{M}^n_2} \leq C2^{-s}||(\Delta u)^s||_{\mathcal{M}^n_2}$$

which leads to the conclusion - remember the first step! - that $\sum_{s\geq 1}(\nabla u)^s \in B^0_{\mathcal{M}^n_2,1}$.
Alternatively one could observe that

$$\Big|\partial^{|\alpha|}\Big(\frac{\xi_i}{|\xi|^2}\Big)\Big| \leq C|\xi|^{-1-|\alpha|}$$

information, which together with theorem 2.9 in [10] leads to the same conclusion as above, namely that

$$\nabla u \in B^0_{\mathcal{M}^n_2,1}.$$

These estimates complete the proof.

□

3.4 Proof of theorem 1.2 iii)

This proof is very similar to the one of theorem 1.2 ii).
In stead of the observation $\Big|\partial^{|\alpha|}\Big(\frac{\xi_i}{|\xi|^2}\Big)\Big| \leq C|\xi|^{-1-|\alpha|}$ here we use theorem 2.9 of [10] together with the fact that

$$\Big|\partial^{|\alpha|}\Big(\frac{\xi_i\xi_j}{|\xi|^2}\Big)\Big| \leq C|\xi|^{-|\alpha|}.$$

□

3.5 Proof of theorem 1.4

Lemma 3.6 *There exist constants $\varepsilon(m) > 0$ and $C(m) > 0$ such that for every $\Omega \in B^0_{\mathcal{M}^n_2,2}(B^n_1(0), so(m) \otimes \Lambda^1\mathbb{R}^n)$ which satisfies*

$$||\Omega|B^0_{\mathcal{M}^n_2,2}|| \leq \varepsilon(m)$$

there exist $\xi \in B^1_{\mathcal{M}^n_2,2}(B^n_1(0), so(m) \otimes \Lambda^{n-2}\mathbb{R}^n)$ and $P \in B^1_{\mathcal{M}^n_2,2}(B^n_1(0), SO(m))$ such that

i)

$$*d\xi = P^{-1}dP + P^{-1}\Omega P \text{ in } B^n_1(0)$$

ii)

$$\xi = 0 \text{ on } \partial B^n_1(0)$$

iii)

$$||\xi|B^1_{\mathcal{M}^n_2,2}|| + ||P|B^1_{\mathcal{M}^n_2,2}|| \leq C(m)||\Omega|B^0_{\mathcal{M}^n_2,2}||.$$

The proof of this lemma is a straightforward adaptation of the corresponding assertion in [16].

Now, let $\varepsilon(m), P$ and ξ be as in lemma 3.6. Note that since $P \in SO(m)$ we have also $P^{-1} \in B^1_{\mathcal{M}^n_2,2}$. Our goal is to find A and B such that

$$dA - A\Omega = -d^*B. \tag{7}$$

If we set $\tilde{A} := AP$ then, according to equation (7) it has to satisfy

$$d\tilde{A} + (d^*B)P = \tilde{A} + d\xi.$$

As a intermediate step we will first study the following problem

$$\begin{cases} \Delta\hat{A} & = d\hat{A} \cdot *d\xi - d^*B \cdot \nabla P \text{ in} B^n_1(0) \\ d(d^*B) & = d\hat{A} \wedge dP^{-1} - d*(\hat{A}d\xi P^{-1}) - d*(d\xi P^{-1}) \\ \frac{\partial\hat{A}}{\partial\nu}, & = 0 \text{ and } B = 0 \text{ on } \partial B^n_1(0) \\ \int_{B^n_1(0)} \hat{A} & = id_m. \end{cases}$$

For this system we have the a-priori-estimates (recall theorem 1.2 with its proof, lemma 2.15 and the fact that we are working on a bounded domain)

$$\begin{aligned} ||\hat{A}|B^1_{\mathcal{M}^n_2,2}|| + ||\hat{A}||_\infty \quad & \leq \quad C||\xi|B^1_{\mathcal{M}^n_2,2}||\ ||\hat{A}|B^1_{\mathcal{M}^n_2,2}|| \\ & \quad\quad +C||P|B^1_{\mathcal{M}^n_2,2}||\ ||B|B^1_{\mathcal{M}^n_2,2}|| \end{aligned}$$

and

$$\begin{aligned} ||B|B^1_{\mathcal{M}^n_2,2}|| \quad & \leq \quad C||P^{-1}|B^1_{\mathcal{M}^n_2,2}||\ ||\hat{A}|B^1_{\mathcal{M}^n_2,2}|| + C||\xi|B^1_{\mathcal{M}^n_2,2}||\ ||\hat{A}||_\infty \\ & \quad\quad +C||\xi|B^1\mathcal{M}^n_2, 2||. \end{aligned}$$

Since the used norms of ξ and P - as well as of P^{-1} - can be bounded in terms of $C||\Omega|B^0_{\mathcal{M}^n_2,2}||$ the above estimates together with standard fixpoint theory

guarantee the existence of $\hat{A}$ and B such that they solve the above system and in addition satisfy

$$||\hat{A}|B^1_{\mathcal{M}^n_2,2}|| + ||\hat{A}||_\infty + ||B|B^1_{\mathcal{M}^m_2,2}|| \leq C||\Omega|B^0_{\mathcal{M}^m_2,2}||. \tag{8}$$

Next, similar to the proof of corollary 1.5 we decompose for some D

$$d\hat{A} - \hat{A} * d\xi + d^*BP = d^*D.$$

Then we set $\tilde{A} := \hat{A} + id_m$, which satisfies for some $n-2$-form F

$$d\tilde{A} - \tilde{A} * d\xi + d^*BP = d^*D - *d\xi =: *dF.$$

It is not difficult to show that $*d(*dFP^{-1}) = 0$ together with $F = 0$ on $\partial B^n_1(0)$ imply that $F \equiv 0$ (see also a similar assertion in [14] and remember that on bounded domains $B^0_{\mathcal{M}^n_2,2} \subset L^2$).
From this we conclude that in fact $\tilde{A}$ satisfies the desired equation. If wet finally set $A := \tilde{A}P^{-1}$ and let B as given in the above system we get that in fact these A and B solve the required relation (7).
So far, we have proved parts ii) and iii) of theorem 1.4 (recall also estimate (8)). Moreover, the invertibility of A follows immediately from its construction, likewise the estimates for ∇A and ∇A^{-1}.
Last but not least, the relation $A = \hat{A}P^{-1} + id_m P^{-1}$ implies that

$$||dist(A, SO(m))||_\infty \leq C||\hat{A}||_\infty \leq C||\Omega|B^0_{\mathcal{M}^n_2,2}||.$$

This completes the proof of theorem 1.4.

□

3.6 Proof of corollary 1.5

The first part of the corollary is a straightforward calculation.
Let A and B be as in theorem 1.4.
Then we have

$$\begin{cases} *d*(Adu) = -d^*B \cdot \nabla u \\ d(Adu) = dA \wedge du. \end{cases}$$

These equations together with a classical Hodge decomposition for Adu

$$Adu = d^*E + dD \text{ with} E, D \in W^{1,2}$$

lead to the following equations

$$\begin{cases} \Delta D = -d^*B \cdot \nabla u \\ \Delta E = dA \wedge du. \end{cases}$$

Since the right hand sides are made of Jacobians we conclude that $D, E \in B^0_{\infty,1}$.
Next, we observe that

$$du = A^{-1}(d^*E + dD) \in B^0_{\mathcal{M}^n_2,1} \subset B^{-1}_{\infty,1}.$$

This holds because $A^{-1} \in B^1_{\mathcal{M}^n_2,2} \cap L^\infty$ (see also theorem 1.4) and $dD, d^*E \in B^0_{\mathcal{M}^n_2,1}$ (see also theorem 1.2 ii)).The proof of the above fact is the same as the

proof of the assertion of lemma 2.15. In a last step we note that (recall the reasons why theorem 1.2 hold) thanks to the information we have so far

$$u \in B^0_{\infty,1} \subset C$$

which completes the proof.

□

3.7 Proof of lemma 2.10

We start with the following observation.
Let $x_0 \in \mathbb{R}^n$ and $r > 0$ and recall that $1 < q \leq 2$ and $r \leq q$. Then for $f \in B^0_{\mathcal{M}^p_q,r}$ we have

$$\begin{aligned}
\Big(\int_{B_r(x_0)} \big(\sum_{s=0}^{\infty} |f^s|^2\big)^{\frac{q}{2}}\Big)^{\frac{1}{q}} &\leq \Big(\int_{B_r(x_0)} \sum_{s=0}^{\infty} |f^s|^q\Big)^{\frac{1}{q}} \\
&\leq \Big(\sum_{s=0}^{\infty} \int_{B_r(x_0)} |f^s|^q\Big)^{\frac{1}{q}} \\
&\leq \Big(\sum_{s=0}^{\infty} ||f^s||^q_{L^q(B_r(x_0))}\Big)^{\frac{1}{q}} \\
&\leq \Big(\sum_{s=0}^{\infty} ||f^s||^q_{\mathcal{M}^p_q} (r^{\frac{n}{q}-\frac{n}{p}})^q\Big)^{\frac{1}{q}} \\
&\leq \Big((r^{\frac{n}{q}-\frac{n}{p}})^q \sum_{s=0}^{\infty} ||f^s||^q_{\mathcal{M}^p_q}\Big)^{\frac{1}{q}} \\
&= r^{\frac{n}{q}-\frac{n}{p}} \Big(\sum_{s=0}^{\infty} ||f^s||^q_{\mathcal{M}^p_q}\Big)^{\frac{1}{q}} \\
&= r^{\frac{n}{q}-\frac{n}{p}} ||f|B^0_{\mathcal{M}^p_q,q}|| \\
&\leq Cr^{\frac{n}{q}-\frac{n}{p}} ||f|B^0_{\mathcal{M}^p_q,r}||.
\end{aligned}$$

From the last inequality we have that for all $r > 0$ and for all $x_0 \in \mathbb{R}^n$

$$r^{\frac{n}{p}-\frac{n}{q}} \Big|\Big| \big(\sum_{s=0}^{\infty} |f^s|^2\big)^{\frac{q}{2}} \Big|\Big|_{L^q(B_r(x_0))} \leq C||f|B^0_{\mathcal{M}^p_q,r}||.$$

This last estimate together [11], proposition 4.1, implies that $f \in \mathcal{M}^p_q$.
The assertion in the case $f \in N^0_{p,q,r}$ is the same.

□

3.8 Proof of lemma 2.12

i) In a first step we will show that if $f \in B^1_{\mathcal{M}^p_q,r}$ there exist a constant C - independent of f - such that

$$||f|B^0_{\mathcal{M}^p_q,r}|| + ||\nabla f|B^0_{\mathcal{M}^p_q,r}|| \leq C||f|B^1_{\mathcal{M}^p_q,r}||.$$

Obviously, we have that

$$||f|B^0_{\mathcal{M}^p_q,r}|| \leq ||f|B^1_{\mathcal{M}^p_q,r}||.$$

Moreover, we observe that

$$\begin{aligned}
||\nabla f|B^0_{\mathcal{M}^p_q,r}|| &= \Big(\sum_{j=0}^{\infty} ||(\nabla f)^j||^r_{\mathcal{M}^q_p}\Big)^{\frac{1}{r}} \\
&\leq \Big(\sum_{j=1}^{\infty} ||(\nabla f)^j||^r_{\mathcal{M}^q_p}\Big)^{\frac{1}{r}} + ||(\nabla f)^0||_{\mathcal{M}^p_q} \\
&\leq C\Big(\sum_{j=1}^{\infty} 2^{jr}||f^j||^r_{\mathcal{M}^q_p}\Big)^{\frac{1}{r}} + C||f||_{\mathcal{M}^p_q} \\
&\quad \text{where for the first addend we used an estimate similar to (3.2)} \\
&\quad\quad \text{with the necessary adaptations to our situation} \\
&\quad\quad \text{see also [10]} \\
&\quad \text{and for the second addend we used [10], lemma 1.8} \\
&\quad\quad \text{and the observation } \mathcal{F}^{-1}(\xi\varphi_0\hat{f}) = \mathcal{F}^{-1}(\xi\varphi_0) * f. \\
&\leq C||f|B^1_{\mathcal{M}^p_q,r}|| + C||f|B^0_{\mathcal{M}^p_q,r}|| \\
&\quad \text{because of lemma 2.10} \\
&\leq C||f|B^1_{\mathcal{M}^p_q,r}|| + C||f|B^1_{\mathcal{M}^p_q,r}|| \\
&\leq ||f|B^1_{\mathcal{M}^p_q,r}||
\end{aligned}$$

as desired.

ii) Now, we assume that f satisfies

$$||f|B^0_{\mathcal{M}^p_q,r}|| + ||\nabla f|B^0_{\mathcal{M}^p_q,r}|| < \infty.$$

We have to show that this last quantity controls

$$||f|B^1_{\mathcal{M}^p_q,r}||.$$

In fact, we calculate

$$\begin{aligned}
||f|B^1_{\mathcal{M}^p_q,r}|| &= \Big(\sum_{j=0}^{\infty} 2^{jr}||f^j||^r_{\mathcal{M}^q_p}\Big)^{\frac{1}{r}} \\
&\leq C||f^0||_{\mathcal{M}^p_q} + C\Big(\sum_{j=1}^{\infty} 2^{jr}||f^j||^r_{\mathcal{M}^q_p}\Big)^{\frac{1}{r}} \\
&\leq C||f^0|B^0_{\mathcal{M}^p_q,r} + C||\nabla f|B^0_{\mathcal{M}^p_q,r}|| \\
&\quad \text{again by an adaption of estimate (3.2)} \\
&\leq C(||f^0|B^0_{\mathcal{M}^p_q,r} + ||\nabla f|B^0_{\mathcal{M}^p_q,r}||).
\end{aligned}$$

□

3.9 Proof of lemma 2.13

According to lemma 2.12 it is enough to show that $f \in B^0_{\mathcal{M}^p_q,r}$. First of all, we observe that

$$\begin{aligned}
\Big|\Big|\sum_{j=1}^{\infty} f^j \Big| B^0_{\mathcal{M}^p_q,r}\Big|\Big| &\leq \Big|\Big|\sum_{j=1}^{\infty} f^j \Big| B^1_{\mathcal{M}^p_q,r}\Big|\Big| \\
&\leq C\Big(\sum_{j=0}^{\infty} 2^{jr}||f^j||^r_{\mathcal{M}^p_q}\Big)^{\frac{1}{r}} \\
&\leq C\Big(\sum_{j=0}^{\infty} ||(\nabla f)^j||^r_{\mathcal{M}^p_q}\Big)^{\frac{1}{r}} \\
&\leq ||\nabla f|B^0_{\mathcal{M}^p_q,r}||.
\end{aligned}$$

Now, it remains to estimate $||f^0||_{\mathcal{M}^p_q}$:
It holds

$$f^0 = \mathcal{F}^{-1}\big(\sum_{i=1}^{n} \frac{\xi_i}{|\xi|^2}\xi_i \hat{f}\varphi_0\big).$$

Next, due to lemma 2.10 and its corollary we know that $f \in L^q$ and in particular - since f has compact support $f \in L^1$ so $\xi_i \hat{f} \in L^\infty$ for all i. Moreover, thanks to our assumptions

$$\varphi_0 \frac{1}{|\xi|} \in L^{\frac{p}{p-1}} \quad \text{where} \quad \frac{p}{p-1} \in [1,2].$$

So, for all possible i

$$\varphi_0 \frac{\xi_i}{|\xi|^2}\xi_i \hat{f} \in L^{\frac{p}{p-1}}.$$

From this we conclude that

$$f^0 \in L^p \subset \mathcal{M}^p_q,$$

and finally

$$\begin{aligned}
||f^0|B^0_{\mathcal{M}^p_q,r}|| &\leq ||f^0||_{\mathcal{M}^p_q} + ||f^1||_{\mathcal{M}^p_q} \\
&\leq ||f^0||_{L^p} + C\Big|\Big|\sum_{j=1}^{\infty} f^j|B^0_{\mathcal{M}^p_q,r}\Big|\Big| \\
&\leq C||\nabla f|B^0_{\mathcal{M}^p_q,r}|| + C\Big|\Big|\sum_{j=1}^{\infty} f^j|B^0_{\mathcal{M}^p_q,r}\Big|\Big| \\
&\leq C||\nabla f|B^0_{\mathcal{M}^p_q,r}||.
\end{aligned}$$

□

3.10 Proof of lemma 2.14

Density of O_M in $N^s_{p,q,r}$ respectively in $B^s_{\mathcal{M}^p_q,r}$
The idea is to approximate $f \in N^s_{p,q,r}$ by $f_n := \sum_{k=0}^{n} f^k$.

From the definition of the spaces $N^s_{p,q,r}$ we immediately deduce that there exists $N \in \mathbb{N}$ such that

$$\Big(\sum_{j=N+1}^{\infty} 2^{sjr}||f^j||^r_{M^p_q}\Big)^{\frac{1}{r}} < \varepsilon.$$

What concerns the first contributions, i.e. f^0-f^N, we know that

$$\sum_{j=0}^{N} f^j =: f_N \in O_M.$$

So,

$$||f - f_N|N^s_{p,q,r}|| \le C\Big(\sum_{j=N+1}^{\infty} 2^{sjr}||f^j||^r_{M^p_q}\Big)^{\frac{1}{r}} < C\varepsilon$$

where C does not depend on f. This shows that f_N approximates f in the desired way.
The proof in the case $B^s_{\mathcal{M}^p_q,r}$ is the same - with the necessary modifications of course.
Density of O_M in $\mathcal{N}^s_{p,q,r}$
The idea is the same as above.
Observe that the definition implies that there exist integers n and m such that

$$\Big(\sum_{j\notin\{-n,\dots,0\dots m\}} 2^{sjr}||f_j||^r_{\mathcal{M}^s_{p,q}}\Big)^{1/r} \le \frac{\varepsilon}{2}.$$

And as before, this gives us the result that O_M is dense in $\mathcal{N}^s_{p,q,r}$.

Another idea to prove the density of C^∞ in $N^s_{p,q,r}$ arises from the usual mollification:
We have to show that for any given ε and any given function $f \in N^s_{p,q,r}$ there exists a function $g \in C^\infty$ such that

$$||f - g|N^s_{p,q,r}|| \le \varepsilon.$$

As indicated above, our candidate for g will be a function of the form

$$g = \varphi_\delta * f$$

where φ_δ is a mollifying sequence (and δ will be specified later on).
First of all, observe that due to Tonelli-Fubini we have $\varphi_\delta * f^j = (\varphi_\delta * f)^j$.
Now, as above we observe that the fact that f belongs to $N^s_{p,q,r}$ implies that there exists $N_0 \in \mathbb{N}$ such that

$$\Big(\sum_{N_0+1}^{\infty} 2^{jsr}||f^j|M^p_q||^r\Big)^{\frac{1}{r}} \le \tilde{\varepsilon}$$

which together with [10], lemma 1.8, immediately leads to the observation that

$$\Big(\sum_{N_0+1}^{\infty} 2^{jsr}||(f - f * \varphi_\delta)^j|M^p_q||^r\Big)^{\frac{1}{r}} \le \frac{\varepsilon}{2}.$$

For the remaining contributions we first of all observe that

$$|f^j - f^j * \varphi_\delta| \leq ||\nabla f^j||_\infty \delta \leq C||f|N^s_{p,q,r}||2^j \delta.$$

In order to see this, note that $f^j \in N^s_{p,q,1}$ which together with two results from [10] similar to the estimate (3.2) and the embedding of Besov-Morrey into Besov spaces (see also [10]) implies that

$$||\nabla f^j||_\infty \leq C||f|N^s_{p,q,r}||2^j.$$

In the case $j = 0$ observe that

$$\begin{aligned}
(\partial_{x_i} f)^0 &= \mathcal{F}^{-1}(i\xi_i \hat{f}\phi_0) \\
&= \mathcal{F}^{-1}(i\xi_i \hat{f}\phi_0(\phi_0 + \phi_1)) \\
&= f^0 * \mathcal{F}^{-1}(i\xi_i(\phi_0 + \phi_1))
\end{aligned}$$

which implies that

$$||\partial_{x_i} f^0|M^p_q|| \leq C||f^0|M^p_q||.$$

Apart from this observation, the argument is the same as the usual one known in the framework of Lebesgue spaces.
Now, we can calculate for any radius $R \in (0, 1]$ and for any point $x_0 \in \mathbb{R}^n$

$$\begin{aligned}
R^{\frac{n}{p}-\frac{n}{q}}||f^j - f^j * \varphi_\delta||_{L^q(B_R(x_0))} &= R^{\frac{n}{p}-\frac{n}{q}}\Big(\int_{B_R(x_0)} |f^j - f^j * \varphi_\delta|^q\Big)^{\frac{1}{q}} \\
&\leq CR^{\frac{n}{p}-\frac{n}{q}}\Big(||\nabla f^j||^q_\infty \delta^q R^n\Big)^{\frac{1}{q}} \\
&\leq CR^{\frac{n}{p}-\frac{n}{q}}\Big(||f|N^s_{p,q,r}||^q 2^{jq}\delta^q R^n\Big)^{\frac{1}{q}} \\
&= CR^{\frac{n}{p}}||f|N^s_{p,q,r}||\delta 2^j \\
&\leq C||f|N^s_{p,q,r}||\delta 2^j
\end{aligned}$$

from which we conclude that

$$\begin{aligned}
\Big(\sum_{j=0}^{N_0} 2^{jsr}||f^j - f^j * \varphi_\delta|M^p_q||^r\Big)^{\frac{1}{r}} &\leq \sum_{j=0}^{N_0} ||f|N^s_{p,q,r}||\delta 2^{N_0+N_0 sr} \\
&\leq (N_0 + 1)||f|N^s_{p,q,r}||\delta 2^{N_0+N_0 sr} \\
&\leq \frac{\varepsilon}{2}
\end{aligned}$$

if we choose δ sufficiently small.
This shows that $f \in N^s_{p,q,r}$ can be approximated by compactly supported smooth function - the convolution $f * \varphi_\delta * f$ has compact support.
Now, we assume that $f \in B^s_{\mathcal{M}^p_q,r}$ where $s \geq 0$, $1 < q \leq 2$ and $1 \leq q \leq p \leq \infty$ has compact support. First of all, we observe that according to lemma 2.10 $f \in \mathcal{M}^p_q$ and since it has compact support, $f \in L^q$. From this we deduce that whenever $O \leq j \leq N_0$, $f^j \in B^s_{q,m}$ for all $s \in \mathbb{R}$ and arbitrary m and in particular, $f^j \in L^p$. So for each j there exists a δ_j such that

$$||f^j - f^j * \varphi_{\delta_j}||^m_q \leq \Big(\frac{\varepsilon}{2(N_0 + 1)}\Big)^m.$$

If we now choose δ small enough, then

$$\Big(\sum_{j=0}^{N_0} 2^{jsr}||f^j - f^j * \varphi_\delta|M_q^p||^r\Big)^{\frac{1}{r}} = \Big(\sum_{j=0}^{N_0} 2^{jsr}||(f - f*)^j\varphi_\delta|M_q^p||^r\Big)^{\frac{1}{r}} \leq \frac{\varepsilon}{2}.$$

The other frequencies are estimated as above.
Finally we observe that $f * \varphi_\delta$ is not only smooth but also compactly supported since it is a convolution of a compactly supported function with a compactly supported distribution.

□

Remark 3.7 A close look at the proof we just gave, shows that in fact

$$\cap_{m\geq 0} C^m$$

is dense in the above spaces.

3.11 Proof of lemma 2.15

We split the product fg into the three paraproducts $\pi_1(f,g)$, $\pi_2(f,g)$ and $\pi_3(f,g)$ and analyse each of them independently.

i) We start with $\pi_1(f,g) = \sum_{k=2}^{\infty}\sum_{l=0}^{k-2} f^l g^k$. It is easy to see that a simple adaptation of lemma 3.15 of [12] to our variant of Besov-Morrey, implies that it suffices to show that

$$\Big(\sum_{k=2}^{\infty} ||g^k \sum_{l=0}^{k-2} f^l||^2_{\mathcal{M}_2^n}\Big)^{\frac{1}{2}} \leq C||g|B^0_{\mathcal{M}_2^n,2}||(||f|B^1_{\mathcal{M}_2^n,2}|| + ||f||_\infty).$$

In fact, we calculate

$$\begin{aligned}
\Big(\sum_{k=2}^{\infty} ||g^k \sum_{l=0}^{k-2} f^l||^2_{\mathcal{M}_2^n}\Big)^{\frac{1}{2}} &\leq \Big(\sum_{k=2}^{\infty} ||g^k (\sup_s |\sum_{l=0}^{s} f^l|)||^2_{\mathcal{M}_2^n}\Big)^{\frac{1}{2}} \\
&\leq \Big(\sum_{k=2}^{\infty} ||g^k||^2_{\mathcal{M}_2^n} || \sup_s |\sum_{l=0}^{s} f^l|||^2_\infty\Big)^{\frac{1}{2}} \\
&\leq || \sup_s |\sum_{l=0}^{s} f^l|||_\infty (\sum_{k=2}^{\infty} ||g^k||^2_{\mathcal{M}_2^n}\Big)^{\frac{1}{2}} \\
&\leq || \sup_s |\sum_{l=0}^{s} f^l|||_\infty ||g|B^0_{\mathcal{M}_2^n,2}|| \\
&\leq ||f||_\infty ||g|B^0_{\mathcal{M}_2^n,2}|| \\
&\quad \text{because of lemma 4.4.2 of [17]} \\
&< \infty.
\end{aligned}$$

ii) Next, we study $\pi_2(f,g) = \sum_{k=0}^{\infty}\sum_{l=k-1}^{k+1} f^l g^k$. For our further calculations we fix $l = k$. We will see that what follows will not depend on this choice, so

$$||\pi_2(f,g)|B^0_{\mathcal{M}_2^n,2}|| \leq C \sup_{s\in\{-1,0,1\}} ||\sum_{k=0}^{\infty} f^{k+s} g^k|B^0_{\mathcal{M}_2^n,2}||.$$

In fact, we will show a bit more, namely $\pi_2(f,g) \in B^1_{\mathcal{M}_1^{\frac{n}{2}},1}$. Again a simple adaptation of lemma 3.16 of [12] shows that we only have to estimate $\sum_{k=0}^{\infty} 2^k ||f^k g^k||_{\mathcal{M}_1^{\frac{n}{2}}}$. In fact, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} 2^k ||f^k g^k||_{\mathcal{M}_1^{\frac{n}{2}}} &\leq \sum_{k=0}^{\infty} 2^k ||f^k||_{\mathcal{M}_2^n} ||g^k||_{\mathcal{M}_2^n} \\
&\leq \Big(\sum_{k=0}^{\infty} 2^{2k} ||g^k||^2_{\mathcal{M}_2^n}\Big)^{\frac{1}{2}} \Big(\sum_{k=0}^{\infty} ||f^k||^2_{\mathcal{M}_2^n}\Big)^{\frac{1}{2}} \\
&\leq ||g|B^1_{\mathcal{M}_2^n,2}||\; ||f|B^0_{\mathcal{M}_2^n,2}|| \\
&< \infty.
\end{aligned}$$

Once we have this, it implies together with the embedding of Besov-Morrey spaces into Besov spaces (see [10]) - adapted to our variant of Besov-Morrey spaces - and the fact that $l^1 \subset l^2$ immediately that $\sum_{k=0}^{\infty} f^k g^k \in B^0_{\mathcal{M}_2^n,2}$. And finally we get that $\pi_2(f,g) \in B^0_{\mathcal{M}_2^n,2}$.

iii) The remaining addend is $\pi_3(f,g)$. Again, as in i) it is enough to show that we can estimate $\Big(\sum_{l=2}^{\infty} ||f^l \sum_{k=0}^{l-2} g^k||^2_{\mathcal{M}_2^n}\Big)^{\frac{1}{2}}$ in the desired manner. In fact we observe that the following inequalities hold:

$$\begin{aligned}
\Big(\sum_{l=2}^{\infty} ||f^l \sum_{k=0}^{l-2} g^k||^2_{\mathcal{M}_2^n}\Big)^{\frac{1}{2}} &\leq \sum_{l=2}^{\infty} ||f^l \sum_{k=0}^{l-2} g^k||_{\mathcal{M}_2^n} \\
&\leq \sum_{l=2}^{\infty} ||f^l||_{\mathcal{M}_2^n} || \sum_{k=0}^{l-2} g^k||_{\infty} \\
&= \sum_{l=2}^{\infty} 2^l ||f^l||_{\mathcal{M}_2^n} 2^{-l} || \sum_{k=0}^{l-2} g^k||_{\infty} \\
&\leq \Big(\sum_{l=0}^{\infty} 2^{2l} ||f^l||^2_{\mathcal{M}_2^n}\Big)^{\frac{1}{2}} \Big(\sum_{l=0}^{\infty} 2^{-2l} || \sum_{k=0}^{l-2} g^k||^2_{\infty}\Big)^{\frac{1}{2}} \\
&\leq C\Big(\sum_{l=0}^{\infty} 2^{2l} ||f^l||^2_{\mathcal{M}_2^n}\Big)^{\frac{1}{2}} \Big(\sum_{l=0}^{\infty} 2^{-2l} || \sum_{k=0}^{l} g^k||^2_{\infty}\Big)^{\frac{1}{2}} \\
&\leq C||f|B^1_{\mathcal{M}_2^n,2}|| \Big(\sum_{l=0}^{\infty} 2^{-2l} || \sum_{k=0}^{l} g^k||^2_{\infty}\Big)^{\frac{1}{2}} \\
&\leq C||f|B^1_{\mathcal{M}_2^n,2}||\; ||g|B^{-1}_{\infty,2}|| \\
&\qquad \text{according to lemma 4.4.2 of [17]} \\
&\leq C||f|B^1_{\mathcal{M}_2^n,2}||\; ||g|N^0_{n,2,2}|| \\
&\qquad \text{due to the embedding result for Besov-Morrey spaces ([10])} \\
&\leq C||f|B^1_{\mathcal{M}_2^n,2}||\; ||g|B^0_{\mathcal{M}_2^n,2}|| \\
&< \infty.
\end{aligned}$$

If we put together all our results from i) to iii) we see that we have the estimate

$$||gf|B^0_{\mathcal{M}_2^n,2}|| \leq C||g|B^0_{\mathcal{M}_2^n,2}||(||f|B^1_{\mathcal{M}_2^n,2}|| + ||f||_{\infty})$$

as claimed.

□

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